LOCAL MONOMIALIZATION OF ANALYTIC MAPS

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Abstract. In this paper local monomialization theorems are proven for morphisms of complex and real analytic spaces. This gives the generalization of the local monomialization theorem for morphisms of algebraic varieties over a field of characteristic zero proven in [17] and [19] to analytic spaces.

1. Introduction

In this paper we prove local monomialization theorems for complex and real analytic morphisms.

A local blow up of an analytic space X is a morphism $\pi : X' \to X$ determined by a triple (U, E, π) where U is an open subset of X, E is a closed analytic subspace of U and π is the composition of the inclusion of U into X with the blowup of E.

Hironaka introduced in his work on analytic sets and maps ([43] and [42]) the notion of an étoile over a complex analytic space X to generalize a valuation of a function field of an algebraic variety. An étoile e over an analytic space X is a subcategory of sequences of local blowups over X which satisfy good properties. If $\pi : X' \to X$ belongs to e, a point $e_{X'} \in X'$, called the center of e on X' is associated to e. The set $\mathcal{E}_X$ of all étoiles over X, with the collection of sets $\mathcal{E}_\pi = \{e \in \mathcal{E}_X \mid \pi \in e\}$ for all $\pi : X' \to X$ which are products of local blow ups as a basis of a topology is the voûte étoilée over X. Hironaka proved that the map $P_X : \mathcal{E}_X \to X$, defined by $P_X(e) = e_X$ is continuous, surjective and proper. The Voûte étoilée can be seen as a generalization of the Zariski Riemann manifold of an algebraic function field, but the comparison is limited. A valuation of a giant field can be associated to an étoile, but this valuation does not enjoy many of the good properties realized by valuations on algebraic function fields ([27]). The basic properties of étoiles are reviewed in Section 3.

Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and that e is an étoile over Y. We prove that φ can be made into a monomial mapping at the center of e after performing sequences of local blowups of nonsingular analytic subvarieties above Y and X. We derive some consequences for complex and real analytic geometry.

Definition 1.1. *Suppose that $\varphi : Y \to X$ is a morphism of complex or real analytic manifolds, and $p \in Y$. We will say that the map φ is monomial at p if there exist regular parameters $x_1, \ldots, x_m, x_{m+1}, \ldots, x_t$ in $\mathcal{O}^{\rm an}_{X,\varphi(p)}$ and $y_1, \ldots, y_n$ in $\mathcal{O}^{\rm an}_{Y,p}$ and $c_{ij} \in \mathbb{N}$ such that*

$$\varphi^*(x_i) = \prod_{j=1}^{n} y_j^{c_{ij}} \text{ for } 1 \le i \le m$$

with $rank(c_{ij}) = m$ and $\varphi^*(x_i) = 0$ for $m < i \le t$.

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There is a related notion of an analytic morphism $\varphi : Y \to X$ being monomial on Y (Definition 3.2).

Our principal result is the following theorem.

Theorem 1.2. *Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and e is an étoile over Y. Then there exists a commutative diagram of complex analytic morphisms*

$$\begin{array}{ccc} Y_e & \overset{\varphi_e}{\to} & X_e \\ \beta\downarrow & & \downarrow\alpha \\ Y & \overset{\varphi}{\to} & X \end{array}$$

such that $\beta \in e$, the morphisms α and β are finite products of local blow ups of nonsingular analytic sub varieties, Y_e and X_e are nonsingular analytic spaces and φ_e is a monomial analytic morphism at the center of e.

There exists a nowhere dense closed analytic subspace F_e of X_e such that $X_e \setminus F_e \to X$ is an open embedding and $\varphi_e^{-1}(F_e)$ is nowhere dense in Y_e.

The last condition on F_e is always true if α, β are sequences of local blow ups and φ is regular (this concept is defined in equation (5)). A regular morphism is the analog in analytic geometry of a dominant morphism in algebraic geometry.

A stronger version of Theorem 1.2 is proven in Theorem 8.12. The analogue of Theorem 1.2 for dominant morphisms of algebraic varieties (over a field of characteristic 0) dominated by a valuation was proven earlier in [17] and [19]. The fact that the theorem is not true in positive characteristic was proven in [26]. It is not difficult to extend the proof of local monomialization along a valuation for dominant morphisms of characteristic zero algebraic varieties to arbitrary (not necessarily dominant) morphisms, using standard theorems from resolution of singularities.

We deduce the following Theorem 1.3 from Theorem 1.2, using the fact that the set of étoiles (La Voûte Étoilée) on a complex analytic space has some good topological properties ([43] and [42]). We use in this and the following theorems stated in this introduction the notion of an analytic morphism $\varphi : Y \to X$ of manifolds being monomial on Y which is defined in Definition 3.2. The proof of Theorem 1.3 is obtained from Theorem 1.2 by utilizing techniques from [44] and [42]. Let K be a compact neighborhood of the point $p \in Y$. Theorem 1.2 produces for each étoile $e \in \mathcal{E}_X$ a morphism $\pi_e : Y_e \to Y$ which lifts the initial morphism $\varphi : Y \to X$ to a morphism $\varphi_e : Y_e \to X_e$ which is monomial at the point e_Y. Since $P_Y : \mathcal{E}_Y \to Y$ is proper, the set $K' = P_Y^{-1}(K')$ is compact. Theorem 1.3 follows by extracting a finite sub cover from an open cover of K' by the preimages of open sets obtained from the Y_e.

Theorem 1.3. *Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and $p \in Y$. Then there exists a finite number t of commutative diagrams of complex analytic morphisms*

$$\begin{array}{ccc} Y_i & \overset{\varphi_i}{\to} & X_i \\ \beta_i\downarrow & & \downarrow\alpha_i \\ Y & \overset{\varphi}{\to} & X \end{array}$$

for $1 \le i \le t$ such that each β_i and α_i are finite products of local blow ups of nonsingular analytic sub varieties, Y_i and X_i are smooth analytic spaces and φ_i is a monomial analytic morphism. Further, there exist compact subsets K_i of Y_i such that $\cup_{i=1}^t \beta_i(K_i)$ is a compact neighborhood of p in Y.

There exist nowhere dense closed analytic subspaces F_i of X_i such that $X_i \setminus F_i \to X$ are open embeddings and $\varphi_i^{-1}(F_i)$ is nowhere dense in Y_i.

A stronger version of Theorem 1.3 is proven in Theorem 8.13 below.

We obtain corresponding theorems for real analytic morphisms.

Theorem 1.4. *Suppose that Y is a real analytic manifold, X is a reduced real analytic space and $\varphi : Y \to X$ is a real analytic morphism. Then there exists a finite number t of commutative diagrams of complex analytic morphisms*

$$\begin{array}{ccc} Y_i & \stackrel{\varphi_i}{\to} & X_i \\ \beta_i \downarrow & & \downarrow \alpha_i \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

for $1 \le i \le t$ such that each β_i and α_i are finite products of local blow ups of nonsingular analytic sub varieties, Y_i and X_i are smooth analytic spaces and φ_i is a monomial analytic morphism. Further, there exist compact subsets K_i of Y_i such that $\cup_{i=1}^t \beta_i(K_i)$ is a compact neighborhood of p in Y.

There exist nowhere dense closed analytic subspaces F_i of X_i such that $X_i \setminus F_i \to X$ are open embeddings and $\varphi_i^{-1}(F_i)$ is nowhere dense in Y_i.

A stronger version of Theorem 1.4 is proven in Theorem 9.7.

An application of Theorem 1.4, showing that Hironaka's rectilinearization theorem can be deduced from local monomialization, is given in [28]. The rectilinearization theorem was first proven by Hironaka in [42]. Different proofs have been given by Denef and Van Den Dries [32] and Bierstone and Milman [11].

Because of the existence of examples such as the Whitney Umbrella, $x^2 - zy^2 = 0$, it is not possible for Theorem 1.4 to hold when Y is only assumed to be a reduced real analytic space. However, using a generalization of the notion of resolution of singularities by Hironaka for real analytic spaces we can generalize Theorem 1.4 to arbitrary reduced analytic spaces.

We recall the definition of a smooth real analytic filtration of a real analytic space.

Definition 1.5. *(Definition 5.8.2* [42]*) Let X be a real analytic space. A smooth real analytic filtration of X is a sequence of closed real analytic subspaces $\{X^i\}_{0 \le i < \infty}$ of X such that*

1) *$X^{(0)} = |X|$ and $X^{(i)} \supset X^{(i+1)}$ for all $i \ge 0$.*
2) *$\{X^{(i)}\}$ is locally finite at every point $p \in X$.*
3) *$X^{(i)} \setminus X^{(i+1)}$ is smooth.*

If X is a reduced real analytic space which is countable at infinity, then X has a smooth real analytic filtration (Proposition 5.8 [42]).

Using resolution of singularities, Hironaka deduces the following result.

Proposition 1.6. *(Desingularization I. (5.10)* [42]*) Suppose that X is a real analytic space and $p \in X$. Then there exists an open neighborhood U of p in X, a finite smooth real analytic filtration $\{U^{(i)}\}$ on U and real analytic morphisms $\pi^{(i)} : \overline{U}^{(i)} \to U^{(i)}$ such that*

1) *Each $\overline{U}^{(i)}$ is smooth and $\pi^{(i)}$ is a sequence of blowups of smooth sub varieties.*
2) *$(\pi^{(i)})^{-1}(U^{(i+1)})$ is nowhere dense in $\overline{U}^{(i)}$ and*
3) *$\pi^{(i)}$ induces an isomorphism $\overline{U}^{(i)} \setminus (\pi^{(i)})^{-1}(U^{(i+1)}) \to U^{(i)} \setminus U^{(i+1)}$.*

In particular, $U = \cup_{i\ge 0}\pi^{(i)}(\overline{U}^{(i)})$.

We deduce the following theorem from Theorem 1.4 and Proposition 1.6.

Theorem 1.7. *Suppose that* $\varphi : Y \to X$ *is a real analytic morphism of reduced real analytic spaces and* $p \in Y$. *Then there exists a finite number* t *of commutative diagrams of real analytic morphisms*

$$\begin{array}{ccc} Y_i & & \\ \beta_i \downarrow & \varphi_i \searrow & \\ Y_i^* & & X_i \\ \gamma_i \downarrow & & \downarrow \alpha_i \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

for $1 \le i \le t$ *such that each* $\gamma_i : Y_i^* \to Y$ *is a resolution of singularities of a component of a smooth real analytic filtration of a neighborhood of* p *in* Y, γ_i, β_i *and* α_i *are finite products of local blow ups of nonsingular analytic sub varieties,* Y_i *and* X_i *are smooth analytic spaces and* φ_i *is a monomial analytic morphism. Further, there exist compact subsets* K_i *of* Y_i *such that* $\cup_{i=1}^t \gamma_i\beta_i(K_i)$ *is a compact neighborhood of* p *in* Y.

There exist nowhere dense closed analytic subspaces F_i *of* X_i *such that* $X_i \setminus F_i \to X$ *are open embeddings and* $\varphi_i^{-1}(F_i)$ *is nowhere dense in* Y_i.

There are a number of local theorems in analytic geometry, including by Hironaka on the local structure of subanalytic sets ([43] and [42]), especially the rectilinearization theorem, by Hironaka, the theorem by Lejeune and Teissier [44] and by Hironaka [42] on local flattening, by Cano on local resolution of 3-dimensional vector fields ([13]), by Denef and van den Dries [32] and Bierstone and Milman ([11]) on the structure of semianalytic and subanalytic sets, by Lichtin ([45], [46])to construct local monomial forms of analytic mappings in low dimensions to prove convergence of series and by Belotto on local resolution and monomialization of foliations ([7]). A global form of the result of [13] holds on an algebraic three fold (over an algebraically closed field of characteristic zero) by combining the theorem of [13] with the patching theorem of Piltant in [51].

For dominant morphisms of algebraic varieties of characteristic zero, local monomialization along an arbitrary valuation is proven in [17] and [19]. It is shown in [26] that local monomialization (and even "weak" local monomialization where the vertical arrows are only required to be birational maps) is not true along an arbitrary valuation in positive characteristic, even for varieties of dimension two.

Global monomialization (toroidalization) has been proven for varieties over algebraically closed fields of characteristic zero for dominant morphisms from a projective 3-fold ([20], [21] and [24]). Weak toroidalization (weak global monomialization), where the vertical arrows giving a toroidal map are only required to be birational is proven globally for algebraic varieties of characteristic zero by Abramovich and Karu [4] and Abramovich, Denef and Karu [5]. Applications of this theorem to quantifier elimination and other important problems in logic are given by Denef in [30] and [31].

The proof of local monomialization in characteristic zero function fields given in [17] and [19] does not readily extend to the case of analytic morphisms. This is because the methods from valuation theory that are used there do not behave well under the infinite extensions of quotient fields of local rings which take place under local blow ups associated to an étoile. The behavior of a valuation associated to an étoile which has rank larger than 1 is particularly wild (examples are given in [27]), and the reduction to rank 1 valuations (the value group is an ordered subgroup of $\mathbb{R}$) in the proofs of [17] and [19] does not

extend to a higher rank valuation which is associated to an étoile. New techniques are developed in this paper which are not sensitive to the rank of a valuation. The notion of "independence of variables" for an étoile, Definition 5.1, replaces the notion of the rational rank of a (rank 1) valuation which is used in [17] and [19]. If e is an étoile over an irreducible complex analytic space X, then we have (as in the classical case of function fields) by Lemma 5.3 [27] the inequalities

$$\mathrm{rank} V_e \le \mathrm{ratrank} V_e \le \dim X$$

where V_e is the valuation ring associated to e.

The proofs of this paper can be adapted to give simpler proofs of the local monomialization theorem for characteristic zero algebraic function fields of [17] and [19]. However, two sources of complexity in the proofs of [17] and [19] do not exist in the case of complex analytic morphisms, and cannot (readily) be eliminated. They are the problem of residue field extension of local rings, and the problem of approximation of formal (analytic) constructions to become algebraic.

The proofs of this paper, and the difficulties which must be overcome are related to the problems which arise in resolution of vector fields and differential forms ([52], [13] , [50], [8]) and in resolution of singularities in positive characteristic (some papers illustrating this are [1], [2], [23], [22], [38], [39], [12], [14], [15], [16]). A common difficulty to monomialization of morphisms, resolution of singularities in positive characteristic and resolution of vector fields is the possibility of a natural order going up after the blow up of an apparently suitable nonsingular sub variety.

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2. A brief overview of the proof

In this section we give an outline of the proof of Theorem 1.2 (and a stronger version, Theorem 8.12). Suppose that $\varphi : Y \to X$ is a complex analytic morphism of complex manifolds and e is an étoile over Y. The first step is to reduce, using Proposition 3.5 in Section 3, to the assumption that φ is quasi regular; that is, if we have a commutative diagram

$$\begin{array}{ccc} Y_1 & \stackrel{\varphi_1}{\to} & X_1 \\ \beta\downarrow & & \downarrow\alpha \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

with $\beta \in e$, and α, β products of local blowups of nonsingular analytic sub varieties then $\varphi_1^* : \mathcal{O}^{\mathrm{an}}_{X_1,\varphi_1(e_{Y_1})} \to \mathcal{O}^{\mathrm{an}}_{Y_1,e_{Y_1}}$ is injective. This proof only uses the statement of the theorem of resolution of singularities. In fact, it is true that if φ is quasi regular then φ is regular (so $\hat\varphi_1^* : \hat{\mathcal{O}}^{\mathrm{an}}_{X_1,\varphi_1(e_{Y_1})} \to \hat{\mathcal{O}}^{\mathrm{an}}_{Y_1,e_{Y_1}}$ is also injective), as can be deduced from the sophisticated local flattening theorem of Hironaka, Lejeune and Teissier [44] and with a different proof by Hironaka in [42]. This deduction is shown in [27]. However, we do not need this for our proof, and in fact deduce it in Corollary 8.11 from our proof. The fact that we only assume quasi regularity, and not regularity, is addressed in the proof of Theorem 1.2 in Proposition 8.5.

With the assumption that φ is quasi regular, we have reduced to the proof of Theorem 8.10, and we have that e induces a restricted étoile on X (as explained at the end of

Section 3). We will also need the fact, explained in Section 3, that there is a valuation ν_e with valuation ring V_e on the union of quotient fields of local rings at the center of e of sequences of local blowups by nonsingular sub varieties above Y which are in e.

The most important types of transformations (sequences of local blow ups or change of variables) used in the proof are the generalized monoidal transformation, GMT and the simple GMT (SGMT), which are defined in Section 5. The full set of transformations used are defined after the proof of Lemma 6.4 in Section 5. A GMT associates to a given set $x_1,\ldots,x_n$ of variables another set $\overline{x}_1,\ldots,\overline{x}_n$ (which are parameters at the point on the corresponding birational extension determined by the étoile e), defined by

$$\overline{x}_i=\prod_{j=1}^{n}(x_j+\alpha_j)^{a_{ij}}$$

where $A=(a_{ij})$ is a matrix of natural numbers with Det $A=\pm1$ and $\alpha_j\in\mathbb{C}$.

A collection of variables $x_1,\ldots,x_n$ is called *independent* if every GMT in $x_1,\ldots,x_n$ is monomial (all $\alpha_j=0$). This is a crucial concept in the proof. A critical fact is that a GMT preserves independence of variables.

In the proof, we inductively construct commutative diagrams

$$\begin{array}{ccc}\tilde{Y} & \stackrel{\tilde{\varphi}}{\to} & \tilde{X}\\ \downarrow & & \downarrow\\ Y & \stackrel{\varphi}{\to} & X\end{array}$$

where the vertical morphisms are products of local blow ups of nonsingular analytic sub varieties which are in e such that there exist regular parameters $x_1,\ldots,x_m$ in $\mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}}$ and $y_1,\ldots,y_n$ in $\mathcal{O}^{\rm an}_{\tilde{Y},e_{\tilde{Y}}}$ such that $y_1,\ldots,y_s$ are independent but $y_1,\ldots,y_s,y_i$ are dependent for all i with $s+1\le i\le n$, $x_1,\ldots,x_r$ are independent, and identifying x_i with $\tilde{\varphi}^*(x_i)$, there is an expression for some l

$$\begin{array}{lll}x_1 & = & y_1^{c_{11}}\cdots y_s^{c_{1s}}\\ & & \vdots\\ x_r & = & y_1^{c_{r1}}\cdots y_s^{c_{rs}}\\ x_{r+1} & = & y_{s+1}\\ & & \vdots\\ x_{r+l} & = & y_{s+l}.\end{array}\tag{1}$$

We necessarily have that $C=(c_{ij})$ has rank r (by Lemma 4.1) with our assumptions. We will say that the variables $(x,y)=(x_1,\ldots,x_m;y_1,\ldots,y_n)$ are *prepared* of type (s,r,l) if all of the above conditions hold. The above diagram (1) is labeled as equation (14) in Section 6, where it is introduced in the proof. We say that $(s_1,r_1,l_1)\ge(s,r,l)$ if $s_1\ge s$, $r_1\ge r$ and $r_1+l_1\ge r+l$, and that $(s_1,r_1,l_1)>(s,r,l)$ if $(s_1,r_1,l_1)\ge(s,r,l)$ and $s_1>s$ or $r_1>r$ or $r_1+l_1>r+l$.

Theorem 8.10, and thus Theorem 1.2, is a consequence of induction using Proposition 8.9, which shows that if $\tilde{\varphi}$ is not monomial, and an expression (1) holds, then we can construct some more local blow ups $Y_1\to\tilde{Y}$ and $X_1\to\tilde{X}$ of nonsingular sub varieties, with $Y_1\to Y\in e$ such that we have a resulting morphism $\varphi_1:Y_1\to X_1$ giving equations (1) with an increase $(s_1,r_1,l_1)>(s,r,l)$.

We will now say a little bit about the proof of Proposition 8.9, and the necessary results preceding it. This is accomplished in Section 8. We start with an expression (1), and

then we perform a sequence of transformations which maintain the form (1) to also put x_{r+l+1} into a monomial form consistent with (1). We may assume that there is no change in (r, s, l) under these transformations (until the very last step), since otherwise we have already obtained a proof of the induction statement.

We make use of the following method to reduce the order of a function along a valuation, taking a Tschirnhaus transformation (Lemma 5.8) and then performing sequences of blow ups to make the coefficients monomials (times units), and then performing a transformation of type 4) (defined after the proof of Lemma 6.4 in Section 5) to get a reduction in multiplicity. This is a variation on the reduction method of Zariski in [53], except we consider valuations of arbitrary rank, and use the Tschirnhaus transformation which was introduced by Abhyankar and developed by Hironaka. This method is used repeatedly through out the proofs.

Another important method is developed in Section 7. We define the notion of a formal series g in $\mathbb{C}[[y_1, \ldots, y_{s+l}]]$ to be *algebraic* over $x_1, \ldots, x_{r+l}$ in Definition 7.1. We consider this notion through the decomposition of a series g expressed in (26) and (26). This decomposition was introduced in [19].

We perform 10 types of transformations to achieve the proof of Proposition 8.9, which are listed after Lemma 6.4. The basic transformations are 1), 2), 4) and 9) which are generalized monoidal transforms, and 3) and 10), which are generally used to make a Tschirnhaus transformation.

In Lemma 8.3, it is shown that we can perform transformations which preserve the form (1) to transform a given element $g \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$ into a monomial in $y_1, \ldots, y_s$ times a unit. The decomposition of Section 7 is essential in the proof of this lemma. From this lemma, we obtain in Lemma 8.4 that if $g \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$ is not algebraic over $x_1, \ldots, x_r$, then we can perform transformations which preserve the form 1 to obtain that

$$g = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s} \tag{2}$$

where P is algebraic over $x_1(1), \ldots, x_{r+l}(1)$ and $y_1(1)^{d_1} \cdots y_s(1)^{d_s}$ is not algebraic over $x_1(1), \ldots, x_r(1)$.

In Proposition 8.5, we show that the natural map of formal power series

$$\mathbb{C}[[x_1, \ldots, x_{r+l+1}]] \to \mathbb{C}[[y_1, \ldots, y_n]]$$

is an inclusion. (Since φ is quasi regular, we must have that the map

$$\mathbb{C}\{\{x_1, \ldots, x_m\}\} \to \mathbb{C}\{\{y_1, \ldots, y_n\}\}$$

is injective.)

Lemmas 8.6 and 8.7 generalize Lemmas 8.3 and 8.4 to the case when $g \in \mathbb{C}\{\{y_1, \ldots, y_n\}\}$.

In Proposition 8.8, we now deduce that there is a sequence of transforms preserving the form (1) such that

$$x_{r+l+1}(1) = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s}$$

with $P \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ algebraic over $x_1(1), \ldots, x_{r+l}(1)$ and $y_1(1)^{d_1} \cdots y_s(1)^{d_s}$ not algebraic over $x_1(1), \ldots, x_r(1)$ or we have an expression

$$x_{r+l+1}(1) = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s} y_{s+l+1}(1)$$

with $P \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ algebraic over $x_1(1), \ldots, x_{r+l}(1)$. It now remains to perform a sequence of transformations which remove the P term. This is accomplished in Proposition 8.9.

3. Preliminaries on analytic maps and étoiles

We require that an analytic space be Hausdorff.

Suppose that X is a complex or real analytic manifold and $p \in X$. Let $K = \mathbb{C}$ or $\mathbb{R}$. Suppose that $x_1, \ldots, x_m$ are regular parameters in $\mathcal{O}^{\rm an}_{X,p}$. Then the completion $\hat{\mathcal{O}}^{\rm an}_{X,p}$ of $\mathcal{O}^{\rm an}_{X,p}$ with respect to its maximal ideal is the ring of formal power series $K[[x_1, x_2, \ldots, x_m]]$. The ring $\mathcal{O}^{\rm an}_{X,p}$ is then identified with the subring $K\{\{x_1, \ldots, x_m\}\}$ of convergent power series. By Abel's theorem, the formal series

$$f = \sum a_{i_1,\ldots,i_m} x_1^{i_1} \cdots x_m^{i_m} \in K[[x_1, \ldots, x_m]]$$

is a convergent power series if and only if there exist positive real numbers $r_1, \ldots, r_m, M$ such that

$$||a_{i_1,\ldots,i_m}|| r_1^{i_1} \cdots r_m^{i_m} \le M \tag{3}$$

for every $i_1, \ldots, i_m$.

The local ring $\mathcal{O}^{\rm an}_{X,p}$ of a point p on a complex or real analytic space X is noetherian and henselian by Theorem 45.5 and fact 43.4 [49]. The local ring $\mathcal{O}^{\rm an}_{X,p}$ is excellent by Section 18 [37] (or Theorem 102, page 291 [48] and by (ii) of Scholie 7.8.3 [37]).

A local blow up of an analytic space X (page 418 [43] or Section 1 [42]) is a morphism $\pi : X' \to X$ determined by a triple (U, E, π) where U is an open subset of X, E is a closed analytic subspace of U and π is the composition of the inclusion of U into X with the blowup of E. If $\pi : X^* \to X$ is a sequence of local blowups, then taking F to be the union of the preimages on X^* of the closed subspaces that are blown up in constructing π, we have that F is a closed analytic subspace of X^* such that the induced morphism $X^* \setminus F \to X$ is an open embedding.

Suppose that X is a real or complex analytic manifold. A divisor E on X is a simple normal crossings (SNC) divisor if the support of E is a union of irreducible smooth codimension 1 sub varieties of X which intersect transversally.

Suppose that $\varphi : Y \to X$ is a morphism of complex or real analytic manifolds. Gabrielov [35] (also [10] for a survey of this and related topics) has defined three ranks of φ at a point q of Y. Let $p = \varphi(q)$. We have induced local homomorphisms

$$\varphi^* : \mathcal{O}^{\rm an}_{X,p} \to \mathcal{O}^{\rm an}_{Y,q}$$

and

$$\hat{\varphi}^* : \hat{\mathcal{O}}^{\rm an}_{X,p} \to \hat{\mathcal{O}}^{\rm an}_{Y,q}$$

on the completions. We define

$$\begin{array}{rcl} r_q(\varphi) & = & \text{generic rank} \\ & = & \text{largest rank of the tangent mapping of } \varphi \text{ in a small open neighborhood of } q, \end{array}$$

$$r_q^{\mathcal{F}}(\varphi) = \dim \hat{\mathcal{O}}^{\rm an}_{X,p}/\mathrm{Kernel}\, \hat{\varphi}^*$$

$$r_q^{\mathcal{A}}(\varphi) = \dim \mathcal{O}^{\rm an}_{X,p}/\mathrm{Kernel}\, \varphi^*.$$

We have

$$r_q(\varphi) \le r_q^{\mathcal{F}}(q) \le r_q^{\mathcal{A}}(\varphi) \le \dim X. \tag{4}$$

We will say that φ is regular at q if all three of these ranks are equal to the dimension of X,

$$r_q(\varphi) = r_q^{\mathcal{F}}(\varphi) = r_q^{\mathcal{A}}(\varphi) = \dim X. \tag{5}$$

If Y is a connected manifold and φ is regular at a point $q \in Y$ then φ is regular everywhere on Y. In this case we will say that φ is regular.

The dimension of a subset E of a complex manifold X at a point $p \in X$ is (page 152 [47])

$$\dim_p E = \sup\{\dim \Gamma \mid \Gamma \text{ is a sub manifold of } U \text{ contained in } E \cap U\ \}$$

where U is a small neighborhood of p in X.

If $\varphi : Y \to X$ is a complex analytic morphism of complex manifolds, $q \in Y$ and $p = \varphi(q)$, then $\dim_p \varphi(U) = r_q(\varphi)$ if U is a sufficiently small neighborhood of q in Y.

If E is a closed analytic subset of the complex manifold X and $p \in E$, then

$$\dim_p E = \dim \mathcal{O}^{\rm an}_{E,p}$$

where $\dim \mathcal{O}^{\rm an}_{E,p}$ is the Krull dimension of the local ring $\mathcal{O}^{\rm an}_{E,p}$.

For real analytic spaces, we use the topological dimension T-$\dim_p$, which is defined analogously (Section 5 of [42]). Rank and dimension are also discussed in [10], along with some illustrative examples.

An étoile is defined in Definition 2.1 [43]. An étoile e over a complex analytic space X is defined as a subcategory of the category of sequences of local blow ups over X.

A sequence of local blow ups of X is the composite of a finite sequence of local blow ups (U_i, E_i, π_i).

Let X be a complex analytic space. $\mathcal{E}(X)$ will denote the category of morphisms $\pi : X' \to X$ which are a sequence of local blow ups. For $\pi_1 : X_1 \to X \in \mathcal{E}(X)$ and $\pi_2 : X_2 \to X \in \mathcal{E}(X)$, $\mathrm{Hom}(\pi_1, \pi_2)$ denotes the X-morphisms $X_2 \to X_1$ (morphisms which factor π_1 and π_2). The set $\mathrm{Hom}(\pi_1, \pi_2)$ has at most one element.

Definition 3.1. *(Definition 2.1* [43]*) Let X be a complex analytic space. An étoile over X is a subcategory e of $\mathcal{E}(X)$ having the following properties:*

1) *If $\pi : X' \to X \in e$ then $X' \neq \emptyset$.*
2) *If $\pi_i \in e$ for $i = 1, 2$, then there exists $\pi_3 \in e$ which dominates π_1 and π_2; that is, $\mathrm{Hom}(\pi_3, \pi_i) \neq 0$ for $i = 1, 2$.*
3) *For all $\pi_1 : X_1 \to X \in e$, there exists $\pi_2 : X_2 \to X \in e$ such that there exists $q \in \mathrm{Hom}(\pi_2, \pi_1)$, and the image $q(X_2)$ is relatively compact in X_1.*
4) *(maximality) If e' is a subcategory of $\mathcal{E}(X)$ that contains e and satisfies the above conditions 1) - 3), then $e' = e$.*

The set of all étoiles over X is denoted by $\mathcal{E}_X$.

Using property 3), Hironaka shows that for $e \in \mathcal{E}_X$, and $\pi : X' \to X \in e$, there exists a uniquely determined point $p_\pi(e) \in X'$ (which we will also denote by $e_{X'}$) which has the property that if $\alpha : Z \to X \in e$ factors as

$$Z \xrightarrow{\beta} X' \xrightarrow{\pi} X,$$

then $\beta(p_\alpha(e)) = p_\pi(e)$. We will also call $e_{X'}$ the center of e on X'.

The étoile associates a point $e_X \in X$ to X and if $\pi_1 : X_1 \to U$ is a local blow up of X such that $e_X \in U$ then $\pi_1 \in e$ and $e_{X_1} \in X_1$ satisfies $\pi_1(e_{X_1}) = e_X$. If $\pi_2 : X_2 \to U_1$ is a local blow up of X_1 such that $e_{X_1} \in U_1$ then $\pi_1\pi_2 \in e$ and $e_{X_2} \in X_2$ satisfies $\pi_2(e_{X_2}) = e_{X_1}$. Continuing in this way, we can construct sequences of local blow ups

$$X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X$$

such that $\pi_1 \cdots \pi_i \in e$, with associated points $e_{X_i} \in X_i$ such that $\pi_i(e_{X_i}) = e_{X_{i-1}}$ for all i.

In Section 5 of [27] it is shown that a valuation can be naturally associated to an étoile. We will summarize this construction here.

Suppose that X is a reduced complex analytic space and e is an étoile over X. We will say that $\pi : X_n \to X \in e$ is nonsingular if π factors as a sequence of local blowups

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

such that X_i is nonsingular for $i \geq 1$. The set of local rings $A_\pi := \mathcal{O}^{\mathrm{an}}_{X_n, e_{X_n}}$ such that π is nonsingular is a directed set, as is the set of quotient fields K_π of the A_π (Lemma 4.3 and Definition 3.2 [27]). Let

$$\Omega_e = \lim_{\rightarrow} K_\pi \text{ and } V_e = \lim_{\rightarrow} A_\pi.$$

Then V_e is a valuation ring of the field Ω_e whose residue field is $\mathbb{C}$ (Lemma 6.1 [27]).

We now summarize some further results from [43]. Let X be a complex analytic space. Let $\mathcal{E}_X$ be the set of all étoiles over X and for $\pi : X_1 \to X$ a product of local blow ups, let

$$\mathcal{E}_\pi = \{e \in \mathcal{E}_X \mid \pi \in e\}. \tag{6}$$

Then the $\mathcal{E}_\pi$ form a basis for a topology on $\mathcal{E}_X$. The space $\mathcal{E}_X$ with this topology is called the voûte étoilée over X (Definition 3.1 [43]). The voûte étoilée is a generalization to complex analytic spaces of the Zariski Riemann manifold of a variety Z in algebraic geometry (Section 17, Chapter VI [54]).

The fields Ω_e are gigantic, while the points of the Zariski Riemann manifold of a variety Z are just (equivalence classes) of valuations of the function field $k(Z)$ of Z, so many of the good properties of valuations of the function field do not hold for the valuation induced by an étoile.

We have a canonical map $P_X : \mathcal{E}_X \to X$ defined by $P_X(e) = e_X$ which is continuous, surjective and proper (Theorem 3.4 [43]). It is shown in Section 2 of [43] that given a product of local blow ups $\pi : X_1 \to X$, there is a natural homeomorphism $j_\pi : \mathcal{E}_{X_1} \to \mathcal{E}_\pi$ giving a commutative diagram

$$\begin{array}{ccccc} \mathcal{E}_{X_1} & \cong \mathcal{E}_\pi & \subset & \mathcal{E}_X \\ P_{X_1} \downarrow & & & \downarrow P_X \\ X_1 & & \stackrel{\pi}{\to} & X. \end{array}$$

Definition 3.2. *Suppose that $\varphi : Y \to X$ is a morphism of complex or real analytic manifolds, and $p \in Y$. We will say that the map φ is monomial at p if there exist regular parameters $x_1, \ldots, x_m, x_{m+1}, \ldots, x_t$ in $\mathcal{O}^{\mathrm{an}}_{X,\varphi(p)}$ and $y_1, \ldots, y_n$ in $\mathcal{O}^{\mathrm{an}}_{Y,p}$ and $c_{ij} \in \mathbb{N}$ such that*

$$\varphi^*(x_i) = \prod_{j=1}^{n} y_j^{c_{ij}} \text{ for } 1 \leq i \leq m$$

with $\mathrm{rank}(c_{ij}) = m$ and $\varphi^*(x_i) = 0$ for $m < i \leq t$. We will say that $y_1 y_2 \cdots y_n = 0$ is a local toroidal structure O at p and that φ is a monomial morphism for the toroidal structure O at p.

We will say that φ is monomial on Y (or simply that φ is monomial) if there exists an open cover of Y by open sets U_k which are isomorphic to open subsets of $\mathbb{C}^n$ (or $\mathbb{R}^n$) and an open cover of X by open sets V_k which are isomorphic to open subsets of $\mathbb{C}^t$ (or $\mathbb{R}^t$) such that $\varphi(U_k) \subset V_k$ for all i and there exist $c_{ij}(k) \in \mathbb{N}$ such that

$$\varphi^*(x_i) = \prod_{j=1}^{n} y_j^{c_{ij}(k)} \text{ for } 1 \leq i \leq m$$

with $rank(c_{ij}) = m$ and $\varphi^*(x_i) = 0$ for $m < i \le t$, and where x_i and y_j are the respective coordinates on $\mathbb{C}^t$ and $\mathbb{C}^n$ (or $\mathbb{R}^t$ and $\mathbb{R}^n$).

We will say that $y_1 y_2 \cdots y_n = 0$ is a local toroidal structure O on U_k and that $\varphi|U_k$ is a monomial morphism for the toroidal structure O on U_k.

Definition 3.3. *Suppose that $\varphi : Y \to X$ is an analytic morphism of connected complex analytic manifolds and e is an étoile over Y. Define*

$$d_e(\varphi) = \min\{r^{\mathcal{A}}_{e_{Y_1}}(\varphi_1)\}$$

where the minimum is over commutative diagrams of analytic morphisms

$$\begin{array}{ccc} Y_1 & \stackrel{\varphi_1}{\to} & X_1 \\ \beta \downarrow & & \downarrow \alpha \\ Y & \stackrel{\varphi}{\to} & X \end{array} \tag{7}$$

such that Y_1 and X_1 are connected complex analytic manifolds, $\beta \in e$, α and β are products of local blowups of nonsingular closed analytic sub varieties and there exists a nowhere dense closed analytic subspace F_1 of X_1 such that $X_1 \setminus F_1 \to X$ is an open embedding and $\varphi_1^{-1}(F_1)$ is nowhere dense in Y_1.

We will say that φ is quasi regular with respect to an étoile e on Y if

$$d_e(\varphi) = r^{\mathcal{A}}_{e_Y}(\varphi) = \dim X.$$

Lemma 3.4. *Suppose that $\varphi : Y \to X$ is a morphism of connected complex analytic manifolds and e is an étoile over Y. Suppose that we have a commutative diagram*

$$\begin{array}{ccc} Y_2 & \stackrel{\varphi_2}{\to} & X_2 \\ \alpha_2 \downarrow & & \downarrow \beta_2 \\ Y_1 & \stackrel{\varphi_1}{\to} & X_1 \\ \alpha \downarrow & & \downarrow \beta \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

such that Y_2, X_2, Y_1 and X_1 are connected complex analytic manifolds, $\alpha \in e$, $\alpha\alpha_2 \in e$ and $\alpha, \alpha_2, \beta, \beta_2$ are products of local blow ups of nonsingular closed analytic sub varieties such that there exists a nowhere dense closed analytic subspace F_2 of X_2 such that $X_2 \setminus F_2 \to X$ is an open embedding and $\varphi_2^{-1}(F_2)$ is nowhere dense in Y_2. Then

$$r^{\mathcal{A}}_{e_{Y_1}}(\varphi_1) \ge r^{\mathcal{A}}_{e_{Y_2}}(\varphi_2).$$

Proof. Let $\mathcal{K}_1$ be the kernel of the homomorphism

$$\varphi_1^* : \mathcal{O}^{\mathrm{an}}_{X_1, \varphi_1(e_{Y_1})} \to \mathcal{O}^{\mathrm{an}}_{Y_1, e_{Y_1}}.$$

The kernel $\mathcal{K}_1$ is a prime ideal. There exists an open neighborhood V of $\varphi_1(e_{Y_1})$ in X_1 such that $\mathcal{K}_1$ is generated by analytic functions $f_1, \ldots, f_r$ on V and $Z_1 = Z(f_1, \ldots, f_r) \subset V$ is analytically irreducible with $\dim_{\varphi_1(e_{Y_1})} Z_1 = r^{\mathcal{A}}_{e_{Y_1}}(\varphi_1)$. We have $e_{Y_1} \in \varphi_1^{-1}(V)$. Let Z_2 be the strict transform of Z_1 in $\beta_2^{-1}(V)$. The open set $\varphi_2^{-1}(\beta_2^{-1}(V)) \not\subset \varphi_2^{-1}(F_2)$ since $\varphi_2^{-1}(F_2)$ is nowhere dense in Y_2 and so $\varphi_2(\alpha_2^{-1}(\varphi_1^{-1}(V))) \not\subset F_2$. But

$$\varphi_2(\varphi_2^{-1}(\beta_2^{-1}(V))) = \varphi_2(\alpha_2^{-1}(\varphi_1^{-1}(V))) \subset \beta_2^{-1}(Z_1)$$

and so $\beta_2^{-1}(Z_1) \not\subset F_2$ and thus $Z_2 \ne \emptyset$, $\varphi_2(\alpha_2^{-1}(\varphi_1^{-1}(V))) \subset Z_2$ and the ideal of the germ of Z_2 at $\varphi_2(e_{Y_2})$ is contained in the kernel $\mathcal{K}_2$ of $\varphi_2^* : \mathcal{O}^{\mathrm{an}}_{X_2, \varphi_2(e_{Y_2})} \to \mathcal{O}^{\mathrm{an}}_{Y_2, e_{Y_2}}$. Thus

$$r^{\mathcal{A}}_{e_{Y_2}}(\varphi_2) \le \dim_{\varphi_2(e_{Y_2})} Z_2 = \dim_{\varphi_1(e_{Y_1})} Z_1 = r^{\mathcal{A}}_{e_{Y_1}}(\varphi_1).$$

$\square$

Proposition 3.5. *Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and $e \in \mathcal{E}_Y$ is an étoile over Y. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} Y_e & \stackrel{\varphi_e}{\to} & X_e \\ \delta\downarrow & & \downarrow\gamma \\ Y & \stackrel{\varphi}{\to} & X \end{array} \tag{8}$$

such that $\delta \in e$, the morphisms γ and δ are finite products of local blow ups of nonsingular analytic sub varieties, Y_e and X_e are smooth analytic spaces, there exists a closed analytic sub manifold Z_e of X_e such that $\varphi_e(Y_e) \subset Z_e$ and the induced analytic map $\varphi_e : Y_e \to Z_e$ is quasi regular with respect to e. Further, there exists a nowhere dense closed analytic subspace F_e of X_e such that $X_e \setminus F_e \to X$ is an open embeddding and $\varphi_e^{-1}(F_e)$ is nowhere dense in Y_e.

Proof. Let

$$\begin{array}{ccc} Y_1 & \stackrel{\varphi_1}{\to} & X_1 \\ \alpha\downarrow & & \downarrow\beta \\ Y & \stackrel{\varphi}{\to} & X \end{array} \tag{9}$$

be a diagram as in (7) such that

$$d_e(\varphi) = r^{\mathcal{A}}_{e_{Y_1}}(\varphi_1).$$

Let $\mathcal{K}$ be the prime ideal which is the kernel of

$$\varphi_1^* : \mathcal{O}^{\mathrm{an}}_{X_1, \varphi_1(e_{Y_1})} \to \mathcal{O}^{\mathrm{an}}_{Y_1, e_{Y_1}}.$$

We can replace X_1 with an open neighborhood V of $\varphi_1(e_{Y_1})$ on which a set of generators of $\mathcal{K}$ are analytic and determine a locally irreducible closed analytic subset Z of V and replace Y_1 with $\varphi_1^{-1}(V)$. After performing an embedded resolution of singularities $X_2 \to X_1$ of Z and a resolution of indeterminacy of the rational map $Y_1 \dashrightarrow X_2$, we may assume that Z is nonsingular. Then we have achieved the conclusions of Proposition 3.5 by Lemma 3.4. $\square$

Suppose that $\varphi : Y \to X$ is a regular morphism of nonsingular complex analytic spaces and that e is an étoile over Y. Then e naturally induces an étoile f over X; we have that $\Omega_f \subset \Omega_e$ and $V_f = V_e \cap \Omega_f$ by Proposition 6.2 [27].

If we do not assume that $\varphi : Y \to X$ is regular, but only that φ is quasi regular with respect to e, then the same construction of an induced étoile on X is valid (by Lemma 3.4 and Proposition 3.5).

We in fact have that a quasi regular morphism is regular, as we deduce in Corollary 8.11. This fact can also be deduced from the local flattening theorem of Hironaka, Lejeune and Teissier [44] and Hironaka [42], as is shown in [27].

4. Valuations on algebraic function fields

We begin this section by reviewing some material from Sections 8,9,10 of [3] and Chapter VI, Section 10 [54].

Let K be an algebraic function field over a field $\overline{k}$, and let ν be a valuation of K which is trivial on $\overline{k}$. Let V_ν be the valuation ring of ν and Γ_ν be the value group of ν. Let

$$0 = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_d \subset V_\nu$$

be the chain of prime ideals in V_ν. Let $U_i = \{\nu(a) \mid a \in \mathfrak{p}_i \setminus \{0\}\}$. Let Γ_i be the complement of U_i and $-U_i$ in Γ_ν. The chain of isolated subgroups in Γ_ν is

$$0 = \Gamma_d \subset \cdots \subset \Gamma_0 = \Gamma_\nu.$$

The valuations composite with ν have the valuation rings $V_{\mathfrak{p}_i}$ with value groups Γ_ν/Γ_i. Let ν_i be the induced valuation ($\nu_i(f)$ is the class of $\nu(f)$ in Γ_ν/Γ_i for $f \in K \setminus \{0\}$). The valuation ν is called zero dimensional if the residue field $V_\nu/\mathfrak{p}_d$ is an algebraic extension of $\overline{k}$. In this section we prove the following lemma. In the case when ν has rank 1 (so there is an order preserving embedding of Γ_ν in $\mathbb{R}$), Lemma 4.1 is proven in Section 9 of [53]. We extend this proof to the case when ν has arbitrary rank d. Related constructions of Perron transforms along a valuation of rank greater than 1 are given by ElHitti in [33].

Lemma 4.1. *Suppose that $\overline{k}$ is a field and ν is a valuation of the quotient field of the polynomial ring $\overline{k}[x_1, \ldots, x_{s+1}]$ such that $\nu(x_i) > 0$ for $1 \le i \le s$, $\nu(x_{s+1}) \ge 0$, $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent and $\nu(x_{s+1})$ is rationally dependent on $\nu(x_1), \ldots, \nu(x_s)$. Then there exists a composition of monoidal transforms (a sequence of blow ups of nonsingular subvarieties) of the form*

$$x_i = \left(\prod_{j=1}^{s} \overline{x}_j^{a_{ij}}\right) \overline{x}_{s+1}^{a_{i,s+1}} \text{ for } 1 \le i \le s \text{ and}$$

$$x_{s+1} = \left(\prod_{j=1}^{s} \overline{x}_j^{a_{s+1,j}}\right) \overline{x}_{s+1}^{a_{s+1,s+1}}$$

such that $\nu(\overline{x}_i) > 0$ for $1 \le i \le s$ and $\nu(\overline{x}_{s+1}) = 0$.

If ν is zero dimensional and $\overline{k}$ is algebraically closed, then there exists $0 \ne \alpha \in \overline{k}$ such that $\nu(\overline{x}_{s+1} - \alpha) > 0$.

Proof. The proof is by decreasing induction on the largest $k \le d$ such that there exist $x_{i_1}, \ldots, x_{i_a}$ (with $1 \le i_1 \le \cdots \le i_a \le s$) such that $\nu(x_{s+1})$ is rationally dependent on $\nu(x_{i_1}), \ldots, \nu(x_{i_a})$ and $\nu(x_{i_1}), \ldots, \nu(x_{i_a}) \in \Gamma_k$. If $k = d$ then $\nu(x_{s+1}) = 0$, and the lemma is trivially satisfied, with (a_{ij}) being the identity matrix.

Suppose that this condition is satisfied for k, and the lemma is true for $k+1$. Without loss of generality, since with this condition we can ignore the variables such that $\nu(x_i) \notin \Gamma_k$, we may assume that $\nu(x_1), \ldots, \nu(x_s) \in \Gamma_k$. After reindexing the x_i, there exists r such that $1 \le r \le s$ and $\nu_{k+1}(x_1), \ldots, \nu_{k+1}(x_r)$ is a basis of the span as a rational vector space of $\nu_{k+1}(x_1), \ldots, \nu_{k+1}(x_s)$ in $(\Gamma_k/\Gamma_{k+1}) \otimes \mathbb{Q}$.

Suppose that there exists t with $r < t \le s$ and $\nu_{k+1}(x_t) \ne 0$. After possibly reindexing $x_{r+1}, \ldots, x_s$ we may assume that $\nu_{k+1}(x_{r+1}) \ne 0$. We necessarily have that $\nu_{k+1}(x_{r+1}) > 0$ since $\nu(x_{r+1}) > 0$. Since Γ_k/Γ_{k+1} is a rank 1 ordered group, we can apply the algorithm of Section 2 on pages 861 - 863 of [53] and Section 9 on page 871 of [53] to construct a sequence of monoidal transforms along ν,

$$x_i = \left(\prod_{j=1}^{r} x_j(1)^{a_{ij}(1)}\right) x_{r+1}(1)^{a_{i,r+1}(1)} \text{ for } 1 \le i \le r \text{ and}$$

$$x_{r+1} = \left(\prod_{j=1}^{r} x_j(1)^{a_{r+1,j}(1)}\right) x_{r+1}(1)^{a_{r+1,r+1}(1)}$$

and $x_i = x_i(1)$ for $r+1 \le i \le s$ such that $\nu_{k+1}(x_i(1)) > 0$ for $1 \le i \le r+1$ and

$$\nu_{k+1}(x_{r+1}(1)) = \lambda_1\nu_{k+1}(x_1(1)) + \cdots + \lambda_r\nu_{k+1}(x_r(1))$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{N}$ (by equation (11') on page 863 [53]). We necessarily have that some $\lambda_i > 0$, so we may assume that $\lambda_1 > 0$. Then perform the sequence of monoidal transforms along ν

$$x_{r+1}(1) = x_1(2)^{\lambda_1 - 1}x_2(2)^{\lambda_2}\cdots x_r(2)^{\lambda_r}x_{r+1}(2)$$

and $x_i(1) = x_i(2)$ for $i \ne r+1$. Then $\nu_{k+1}(x_i(2)) > 0$ for all i with $1 \le i \le r+1$ and $\nu_{k+1}(x_{r+1}(2)) = \nu_{k+1}(x_1(2))$. We necessarily have that

$$\nu\left(\frac{x_1(2)}{x_{r+1}(2)}\right) > 0 \text{ or } \nu\left(\frac{x_{r+1}(2)}{x_1(2)}\right) > 0$$

as $\nu(x_1(2)), \ldots, \nu(x_s(2))$ are rationally independent. In the first case, perform the monoidal transform along ν

$$x_1(2) = x_1(3)x_{r+1}(3),\ x_{r+1}(2) = x_1(3) \text{ and } x_i(2) = x_i(3) \text{ for } i \ne 1 \text{ or } r+1.$$

Otherwise, perform the monoidal transform along ν

$$x_1(2) = x_1(3),\ x_{r+1}(2) = x_1(3)x_{r+1}(3) \text{ and } x_i(2) = x_i(3) \text{ for } i \ne 1 \text{ or } r+1.$$

We then have that $\nu(x_i(3)) > 0$ for $1 \le i \le s+1$, $\nu_{k+1}(x_1(3)), \ldots, \nu_{k+1}(x_r(3))$ is a rational basis of the span of $\nu_{k+1}(x_1(3)), \ldots, \nu_{k+1}(x_s(3))$ as a rational vector space in $(\Gamma_k/\Gamma_{k+1}) \otimes \mathbb{Q}$, $\nu(x_1(3)), \ldots, \nu(x_s(3))$ are rationally independent, and $\nu(x_{s+1}(3))$ is rationally dependent on $\nu(x_1(3)), \ldots, \nu(x_s(3))$. We further have that $\nu_{k+1}(x_{r+1}(3)) = 0$. We repeat this algorithm, reducing to the case that $\nu_{k+1}(x_i) = 0$ if $r+1 \le i \le s$.

Suppose that $\nu_{k+1}(x_{s+1}) > 0$ (and $\nu_{k+1}(x_i) = 0$ for $r+1 \le i \le s$). Then we apply the algorithm that we used above to construct a monoidal transform along ν

$$\begin{aligned} x_i &= \left(\textstyle\prod_{j=1}^{r} x_j(1)^{a_{ij}(1)}\right) x_{s+1}(1)^{a_{i,r+1}(1)} \text{ for } 1 \le i \le r \text{ and} \\ x_{s+1} &= \left(\textstyle\prod_{j=1}^{r} x_j(1)^{a_{r+1,j}(1)}\right) x_{s+1}(1)^{a_{r+1,r+1}(1)} \end{aligned} \tag{10}$$

to achieve $\nu_{k+1}(x_i(1)) > 0$ for $1 \le i \le r$, $\nu_{k+1}(x_{s+1}(1)) = 0$ and $\nu(x_{s+1}(1)) \ge 0$. Since $\nu_{k+1}(x_1), \ldots, \nu_{k+1}(x_r)$ are rationally independent, (10) implies that $\nu_{k+1}(x_1(1)), \ldots, \nu_{k+1}(x_r(1))$ are rationally independent. Since $\nu_{k+1}(x_i) = 0$ for $r < i \le s$ and $\nu(x_{r+1}), \ldots, \nu(x_s) \in \Gamma_{k+1}$ are rationally independent we have that

$$\nu(x_1(1)), \ldots, \nu(x_r(1)), \nu(x_{r+1}), \ldots, \nu(x_s)$$

are rationally independent. Since

$$\nu(x_1(1)), \ldots, \nu(x_r(1)), \nu(x_{r+1}), \ldots, \nu(x_s), \nu(x_{s+1}(1))$$

and $\nu(x_1), \ldots, \nu(x_s)$ span the same rational subspace V of $\Gamma_\nu \otimes \mathbb{Q}$, which has dimension s, we have that

$$\nu(x_1(1)), \ldots, \nu(x_r(1)), \nu(x_{r+1}), \ldots, \nu(x_s)$$

is a rational basis of V, so $\nu(x_{s+1}(1))$ is a rational linear combination of

$$\nu(x_1(1)), \ldots, \nu(x_r(1)), \nu(x_{r+1}), \ldots, \nu(x_s).$$

Since $\nu_{k+1}(x_{s+1}(1)) = 0$ and $\nu(x_{k+1}(1)), \ldots, \nu_{k+1}(x_r(1))$ are rationally independent, we have that $\nu(x_{s+1}(1))$ is a rational linear combination of $\nu(x_{r+1}), \ldots, \nu(x_s) \in \Gamma_{k+1}$. We thus attain the conclusions of the lemma by decreasing induction on k.

Finally, if ν is zero dimensional and $\overline{k}$ is algebraically closed, then the class α of $\overline{x}_{s+1}$ in the residue field $\overline{k}$ of V_ν is nonzero. Then necessarily $\nu(\overline{x}_{s+1} - \alpha) > 0$.

□

5. Generalized Monoidal Transforms

Suppose that X is a nonsingular complex analytic space and e is an étoile over X. Let ν_e be a valuation of Ω_e whose valuation ring is V_e (Section 3). Suppose that $\tilde{X} \to X \in e$ and $x_1, \ldots, x_n$ is a regular system of parameters in $\mathcal{O}^{\mathrm{an}}_{\tilde{X}, e_{\tilde{X}}}$. Suppose that $\overline{X} \to \tilde{X}$ is such that $\overline{X} \to \tilde{X} \to X \in e$. The germ of the local homomorphism $\mathcal{O}^{\mathrm{an}}_{\tilde{X}, e_{\tilde{X}}} \to \mathcal{O}^{\mathrm{an}}_{\overline{X}, e_{\overline{X}}}$ is a Generalized Monoidal Transform (GMT) along the étoile e if $\mathcal{O}^{\mathrm{an}}_{\overline{X}, e_{\overline{X}}}$ has regular parameters $\overline{x}_1, \ldots, \overline{x}_n$ such that there exists an $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \in \mathbb{N}$ and $\mathrm{Det}(A) = \pm 1$ such that

$$x_i = \prod_{j=1}^{n} (\overline{x}_j + \alpha_j)^{a_{ij}} \tag{11}$$

for $1 \le j \le n$ and $\alpha_j \in \mathbb{C}$ (at least one of which must be zero since $\mathcal{O}^{\mathrm{an}}_{\tilde{X}, e_{\tilde{X}}} \to \mathcal{O}^{\mathrm{an}}_{\overline{X}, e_{\overline{X}}}$ is a local homomorphism). We will say that the GMT is in the variables $x_{i_1}, \ldots, x_{i_m}$ if the GMT has the special form

$$x_i = \prod_{j \in S} (\overline{x}_j + \alpha_j)^{a_{ij}}$$

for $i \in S$ and

$$x_i = \overline{x}_i$$

for $i \notin S$ where $S = \{i_1, \ldots, i_m\}$. We will say that the GMT is monomial if all α_j are zero. We observe that a GMT is a regular morphism.

It will be assumed through out this paper that all GMT are along a fixed étoile e.

Definition 5.1. *The variables $x_1, \ldots, x_s$ are said to be dependent if there exists a GMT (11) in $x_1, \ldots, x_s$ which is not monomial.*

The variables $x_1, \ldots, x_s$ are said to be independent if they are not dependent.

Lemma 5.2. *Suppose that $x_1, \ldots, x_s$ are independent and (11) is a GMT in $x_1, \ldots, x_s$. Then $\overline{x}_1, \ldots, \overline{x}_s$ are independent.*

Proof. This follows since a composition of a GMT in $x_1, \ldots, x_s$ and in $\overline{x}_1, \ldots, \overline{x}_s$ is a GMT in $x_1, \ldots, x_s$. □

Definition 5.3. *A GMT is a simple GMT (SGMT) if it can be factored by a sequence of blow ups of nonsingular subvarieties.*

Lemma 5.4. *The variables $x_1, \ldots, x_s$ are independent if and only if every SGMT in $x_1, \ldots, x_s$ is monomial.*

Proof. Suppose that every SGMT in $x_1, \ldots, x_s$ is monomial and (11) is a GMT in $x_1, \ldots, x_s$. We must show that all $\alpha_i = 0$. Let ν be the valuation of the quotient field K of $\mathbb{C}[x_1, \ldots, x_s]$ which gives the restriction of ν_e to K. Let $\pi : Z \to \mathbb{A}^s$ be a projective morphism of non-singular toric varieties such that $\overline{x}_1, \ldots, \overline{x}_s$ are regular parameters in $\mathcal{O}_{Z,p}$, where p is the

center of ν on Z. Let J be a (monomial) ideal in $\mathbb{C}[x_1,\ldots,x_s]$ whose blow up in $\mathbb{A}^s$ is Z. By principalization of ideals (a particularly simple algorithm which is adequate for our purposes is given in [36]), there exists a projective morphism of nonsingular toric varieties $\Lambda : Z_1 \rightarrow \mathbb{A}^s$ which is a product of blow ups of nonsingular varieties such that $J\mathcal{O}_{Z_1}$ is locally principal, and so Λ factors through π. Let I be a monomial ideal such that Z_1 is the blow up of I.

Let X_1 be obtained by blowing up I in a neighborhood of $e_{\tilde{X}}$ in $\tilde{X}$. Then $\mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}} \rightarrow \mathcal{O}^{\rm an}_{X_1,e_{X_1}}$ is a SGMT (since $Z_1 \rightarrow \mathbb{A}^s$ is a morphism of toric varieties which is a product of blow ups of nonsingular varieties). Thus $\mathcal{O}_{Z_1,p_1}$ has regular parameters $\tilde{x}_1,\ldots,\tilde{x}_s$ (where p_1 is the center of ν on Z_1) and $\tilde{x}_1,\ldots,\tilde{x}_s,x_{s+1},\ldots,x_n$ are regular parameters in $\mathcal{O}^{\rm an}_{X_1,e_{X_1}}$ such that $x_i = \prod_{j=1}^s \tilde{x}_j^{b_{ij}}$ are monomials for $1 \le i \le s$. Since Λ factors through π, and so there is a factorization

$$\mathcal{O}_{\tilde{X},e_{\tilde{X}}} \rightarrow \mathcal{O}_{\overline{X},\mathcal{O}_{\overline{X}}} \rightarrow \mathcal{O}_{X_1,e_{X_1}}$$

we must also have that the given GMT (11) is monomial. $\square$

Lemma 5.5. *Suppose that $x_1,\ldots,x_s$ are independent and*

$$M_1 = x_1^{d_1(1)}\cdots x_s^{d_s(1)},\ M_2 = x_1^{d_1(2)}\cdots x_s^{d_s(2)}$$

are monomials with $d_i(j) \in \mathbb{N}$. Then there exists a (monomial) SGMT in $x_1,\ldots,x_s$ such that the ideal generated by M_1 and M_2 is principal in $\mathcal{O}^{\rm an}_{X_1,e_{X_1}}$.

Proof. Let ν be the valuation of the quotient field K of $\mathbb{C}[x_1,\ldots,x_s]$ which gives the restriction of ν_e to K. Since $x_1,\ldots,x_s$ are independent, $\nu(x_1),\ldots,\nu(x_s)$ are rationally independent by Lemma 4.1. Let I be the ideal generated by M_1 and M_2 in $\mathbb{C}[x_1,\ldots,x_s]$ There exists a birational morphism of nonsingular toric varieties which is a product of blow ups of nonsingular subvarieties $\pi : Z \rightarrow \mathbb{A}^s$ such that $I\mathcal{O}_Z$ is an invertible ideal sheaf. Let p_1 be the center of ν on Z. Since π is toric and $\nu(x_1),\ldots,\nu(x_s)$ are rationally independent, there exist regular parameters $\overline{x}_1,\ldots,\overline{x}_s$ in $\mathcal{O}_{Z,p_1}$ such that

$$x_i = \prod_{j=1}^s \overline{x}_j^{a_{ij}} \tag{12}$$

for $1 \le i \le s$ are monomials in $\overline{x}_1,\ldots,\overline{x}_s$. Let J be the monomial ideal in $\mathbb{C}[x_1,\ldots,x_s]$ whose blow up is Z. Let X_1 be the blow up of J in a neighborhood of $e_{\tilde{X}}$ in $\tilde{X}$. Then $\overline{x}_1,\ldots,\overline{x}_s,x_{s+1},\ldots,x_m$ are regular parameters in $\mathcal{O}^{\rm an}_{X_1,e_{X_1}}$ and $I\mathcal{O}^{\rm an}_{X_1,e_{X_1}}$ is a principal ideal. $\square$

Lemma 5.6. *Suppose that $x_1,\ldots,x_s \in \mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}}$ are independent, $\gamma \in \mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}}$ is a unit and $d_1,\ldots,d_s \in \mathbb{Q}$. Then $\tilde{x}_1 = \gamma^{d_1}x_1,\ldots,\tilde{x}_s = \gamma^{d_s}x_s$ are independent.*

Proof. Suppose that $\tilde{x}_1,\ldots,\tilde{x}_s$ are not independent. Then there exists $\overline{X} \rightarrow \tilde{X}$ giving a GMT $\tilde{x}_i = \prod_{j=1}^s(\hat{x}_j+\hat{\alpha}_j)^{a_{ij}}$ for $1 \le j \le s$ with some $\hat{\alpha}_j \ne 0$. After reindexing the $\tilde{x}_i$, we may assume that $\hat{\alpha}_j = 0$ for $1 \le j \le a < s$ and $\hat{\alpha}_j \ne 0$ for $a < j \le s$. Define $c_1,\ldots,c_s \in \mathbb{Q}$ by

$$\begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix}$$

where $A=(a_{ij})$. Then $\prod_{j=1}^{s}(\gamma^{c_{ij}})^{a_{ij}}=\gamma^{d_i}$ for $1\le i\le s$. We have

$$\gamma^{c_j}\equiv\gamma(0)^{c_j} \bmod (\hat{x}_1,\ldots,\hat{x}_a)\mathcal{O}^{\mathrm{an}}_{\overline{x},e_{\overline{x}}} \text{ for all } j. \tag{13}$$

Set $\overline{x}_j=\gamma^{c_j}\hat{x}_j$ for $1\le j\le a$, and define $\alpha_j=\gamma(0)^{c_j}\hat{\alpha}_j$, $\overline{x}_j=\gamma^{c_j}(\hat{x}_j+\hat{\alpha}_j)-\alpha_j$ for $a\le j\le s$. Then $\overline{x}_1,\ldots,\overline{x}_s$ are regular parameters in $\mathcal{O}^{\mathrm{an}}_{\overline{X},e_{\overline{X}}}$ by (13). Thus we have a GMT

$$x_i=\prod_{j=1}^{s}(\overline{x}_j+\alpha_j)^{a_{ij}} \text{ for } 1\le j\le s$$

in $x_1,\ldots,x_s$, contradicting the independence of $x_1,\ldots,x_s$ since some $\alpha_j\ne 0$. ☐

Lemma 5.7. *Suppose that $x_1,\ldots,x_s$ are independent and $x_1,\ldots,x_s,x_{s+1}$ are dependent. Suppose that (11) is A GMT in $x_1,\ldots,x_{s+1}$ such that some $\alpha_j\ne 0$. Then there are $x_1(1),\ldots,x_{s+1}(1)$ in $\mathcal{O}^{\mathrm{an}}_{\overline{X},e_{\overline{X}}}$ such that $x_1(1),\ldots,x_{s+1}(1),x_{s+2},\ldots,x_n$ are a regular system of parameters in $\mathcal{O}^{\mathrm{an}}_{\overline{X},e_{\overline{X}}}$ and there is an expression*

$$x_i=\prod_{j=1}^{s}x_j(1)^{b_{ij}} \text{ for } 1\le i\le s$$

and

$$x_{s+1}=\prod_{j=1}^{s}x_j(1)^{b_j}(x_{s+1}(1)+\alpha)$$

where $0\ne\alpha\in\mathbb{C}$, $b_{ij},b_j\in\mathbb{N}$ and the $s\times s$ matrix (b_{ij}) has nonzero determinant. Further, the variables $x_1(1),\ldots,x_s(1)$ are independent.

Proof. Let $R=\mathbb{C}[x_1,\ldots,x_{s+1}]_{(x_1,\ldots,x_{s+1})}$ and K be the quotient field of R. Let (11) be a GMT in $x_1,\ldots,x_s,x_{s+1}$ which is not monomial and $R_1=\mathbb{C}[\overline{x}_1,\ldots,\overline{x}_{s+1}]_{(\overline{x}_1,\ldots,\overline{x}_{s+1})}$. We have a commutative diagram of injective local homomorphisms

$$\begin{array}{ccc} R & \rightarrow & \mathcal{O}^{\mathrm{an}}_{\tilde{X},e_{\tilde{X}}}\\ \downarrow & & \downarrow\\ R_1 & \rightarrow & \mathcal{O}^{\mathrm{an}}_{\overline{X},e_{\overline{X}}}. \end{array}$$

The field K is also the quotient field of R_1 and $R\to R_1$ is birational. Let ν be the restriction of ν_e to K. We have that ν dominates R and ν dominates R_1. Since all GMT in $x_1,\ldots,x_s$ are monomial, we must have that $\nu(x_1),\ldots,\nu(x_s)$ are rationally independent by Lemma 4.1. We have that

$$\nu(x_i)=\sum_{j=1}^{s+1}a_{ij}\nu(\overline{x}_j+\alpha_j) \text{ for } 1\le i\le s+1.$$

Thus after possibly interchanging the variables $\overline{x}_1,\ldots,\overline{x}_{s+1}$, we have that $\alpha_1=\ldots=\alpha_s=0$. Further, since our GMT (11) is not monomial, we must have that $\alpha_{s+1}\ne 0$. Thus the $s\times s$ matrix consisting of the first s rows and columns of $A=(a_{ij})$ has rank s and $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_s)$ are rationally independent. There exists $\lambda_i\in\mathbb{Q}$ such that after replacing $\overline{x}_i$ with $x_i(1):=(\overline{x}_{s+1}+\alpha_{s+1})^{\lambda_i}\overline{x}_i$ for $1\le i\le s$, we have that $x_i=\prod_{j=1}^{s}x_j(1)^{a_{ij}}$ for $1\le i\le s$ and $x_{s+1}=\prod_{j=1}^{s}x_j(1)^{a_{s+1,j}}(\overline{x}_{s+1}+\alpha_{s+1})^{\lambda}$ where $\lambda\in\mathbb{Q}$ is non zero since $\mathrm{Det}(A)\ne 0$. Setting $x_{s+1}(1):=(\overline{x}_{s+1}+\alpha_{s+1})^{\lambda}-\alpha_{s+1}^{\lambda}$ and $\alpha=\alpha_{s+1}^{\lambda}$, we obtain the expression of the GMT asserted in the lemma.

The values $\nu_e(\overline{x}_1),\ldots,\nu_e(\overline{x}_s)$ are rationally independent, and $\nu_e(\overline{x}_{s+1}+\alpha_{s+1})=0$, so $\nu_e(x_1(1)),\ldots,\nu_e(x_s(1))$ are rationally independent. Thus $x_1(1),\ldots,x_s(1)$ are independent. $\square$

The following lemma giving a Tschirnhaus transformation will be useful.

Lemma 5.8. *Suppose that $F\in\mathbb{C}\{\{x_1,\ldots,x_n\}\}$ and ord $F(0,\ldots,0,x_n)=t\geq 1$. Then there exists $\Phi\in\mathbb{C}\{\{x_1,\ldots,x_{n-1}\}\}$ such that setting $\overline{x}_n=x_n-\Phi$, we have that*

$$F=\tau_0\overline{x}_n^t+\tau_2\overline{x}_n^{t-2}+\cdots+\tau_t$$

where $\tau_0\in\mathbb{C}\{\{x_1,\ldots,\overline{x}_n\}\}$ is a unit and $\tau_i\in\mathbb{C}\{\{x_1,\ldots,x_{n-1}\}\}$ for $2\leq i\leq t$.

Proof. By the implicit function theorem (cf. Section C.2.4 [47]),

$$\frac{\partial^{t-1}F}{\partial x_n^{t-1}}=u(x_n-\Phi)$$

where $u\in\mathbb{C}\{\{x_1,\ldots,x_n\}\}$ is a unit series and $\Phi\in\mathbb{C}\{\{x_1,\ldots,x_{n-1}\}\}$. Let $\overline{x}_n=x_n-\Phi$. Let $G(x_1,\ldots,x_{n-1},\overline{x}_n)=F(x_1,\ldots,x_n)$. We expand

$$\begin{array}{lll} G & = & G(x_1,\ldots,x_{n-1},0)+\frac{\partial G}{\partial\overline{x}_n}(x_1,\ldots,x_{n-1},0)\overline{x}_n+\cdots+\frac{1}{(t-1)!}\frac{\partial^{t-1}G}{\partial\overline{x}_n^{t-1}}(x_1,\ldots,x_{n-1},0)\overline{x}_n^{t-1}\\ & & +\frac{1}{t!}\frac{\partial^t G}{\partial\overline{x}_n^t}(x_1,\ldots,x_{n-1},0)\overline{x}_n^t+\cdots \end{array}$$

We have

$$\frac{\partial^{t-1}G}{\partial\overline{x}_n^{t-1}}(x_1,\ldots,x_{n-1},0)=\frac{\partial^{t-1}F}{\partial x_n^{t-1}}(x_1,\ldots,x_{n-1},\Phi)=0$$

and

$$\frac{\partial^t G}{\partial\overline{x}_n^t}(x_1,\ldots,x_{n-1},0)=\frac{\partial^t F}{\partial x_n^t}(x_1,\ldots,x_{n-1},\Phi)$$

is a unit in $\mathbb{C}\{\{x_1,\ldots,x_n\}\}$, giving (by (3)) the conclusions of the lemma. $\square$

6. Transformations

Suppose that $\varphi:Y\to X$ is an analytic morphism of complex analytic manifolds and e is an étoile over Y such that φ is quasi regular with respect to e (Section 3). We will also denote the induced étoile on X (Section 3) by e.

Suppose that $\tilde{Y}\to Y\in e$ and $\tilde{X}\to X\in e$ give a morphism $\tilde{\varphi}:\tilde{Y}\to\tilde{X}$. Then

$$\tilde{\varphi}^*:\mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}}\to\mathcal{O}^{\rm an}_{\tilde{Y},e_{\tilde{Y}}}$$

is injective, so we may regard $\mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}}$ as a subring of $\mathcal{O}^{\rm an}_{\tilde{Y},e_{\tilde{Y}}}$. Assume that there exist regular parameters $x_1,\ldots,x_m$ in $\mathcal{O}^{\rm an}_{\tilde{X},e_{\tilde{X}}}$ and $y_1,\ldots,y_n$ in $\mathcal{O}^{\rm an}_{\tilde{Y},e_{\tilde{Y}}}$ such that $y_1,\ldots,y_s$ are independent but $y_1,\ldots,y_s,y_i$ are dependent for all i with $s+1\leq i\leq n$, $x_1,\ldots,x_r$ are independent, and identifying x_i with $\tilde{\varphi}^*(x_i)$, there is an expression for some l

$$\begin{array}{lll} x_1 & = & y_1^{c_{11}}\cdots y_s^{c_{1s}}\\ & \vdots & \\ x_r & = & y_1^{c_{r1}}\cdots y_s^{c_{rs}}\\ x_{r+1} & = & y_{s+1}\\ & \vdots & \\ x_{r+l} & = & y_{s+l}. \end{array}\tag{14}$$

We necessarily have that $C=(c_{ij})$ has rank r (by Lemma 4.1) with our assumptions, and so by the rank theorem (page 134 [47]) and the inequality (4) there is an induced inclusion

$$\mathbb{C}[[x_1,\ldots,x_{r+l}]]\rightarrow\mathbb{C}[[y_1,\ldots,y_n]].$$

Assume that E_Y is a SNC divisor on Y supported on $Z(y_1y_2\cdots y_s)$ (in a neighborhood of e_Y) in Y.

Definition 6.1. *We will say that the variables $(x,y)=(x_1,\ldots,x_m;y_1,\ldots,y_n)$ are prepared of type (s,r,l) if all of the above conditions hold.*

We will say that $(s_1,r_1,l_1)\ge(s,r,l)$ if $s_1\ge s$, $r_1\ge r$ and $r_1+l_1\ge r+l$, and that $(s_1,r_1,l_1)>(s,r,l)$ if $(s_1,r_1,l_1)\ge(s,r,l)$ and $s_1>s$ or $r_1>r$ or $r_1+l_1>r+l$.

We will perform transformations of the types 1) - 10) below, which preserve the form (14) (and the quasi regularity of the morphism of germs), giving an expression

$$\begin{array}{rcl}x_1(1)&=&y_1(1)^{c_{11}(1)}\cdots y_s(1)^{c_{1s}(1)}\\&\vdots&\\x_r(1)&=&y_1(1)^{c_{r1}(1)}\cdots y_s(1)^{c_{rs}(1)}\\x_{r+1}(1)&=&y_{s+1}(1)\\&\vdots&\\x_{r+l}(1)&=&y_{s+l}(1)\end{array}\tag{15}$$

where $x_1(1),\ldots,x_m(1)$ and $y_1(1),\ldots,x_n(1)$ are respective regular parameters in $\mathcal{O}^{\rm an}_{\overline{X},e_{\overline{X}}}$ and $\mathcal{O}^{\rm an}_{\overline{Y},e_{\overline{Y}}}$ in the induced commutative diagram of quasi regular analytic morphisms

$$\begin{array}{ccc}\overline{Y}&\xrightarrow{\overline{\varphi}}&\overline{X}\\\downarrow&&\downarrow\\\tilde{Y}&\xrightarrow{\tilde{\varphi}}&\tilde{X}.\end{array}$$

where $\overline{Y}\rightarrow\tilde{Y}\rightarrow Y\in e$ and $\overline{X}\rightarrow\tilde{X}\rightarrow X\in e$.

Further, we will have that $x_1(1),\ldots,x_r(1)$ are independent and $y_1(1),\ldots,y_s(1)$ are independent. So we either continue to have that $y_1(1),\ldots,y_s(1),y_t(1)$ are dependent for all $s+1\le t\le n$ or after rewriting (14), we have an increase in s, without decreasing r or $r+l$. In summary, we will have that the variables $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1)\ge(s,r,l)$.

Let $E_{\overline{Y}}$ be the pullback of E_Y on $\overline{Y}$. Then

$$\begin{array}{l}E_{\overline{Y}}\text{ is supported on }Z(y_1(1)y_2(1)\cdots y_s(1))\subset\overline{Y}\\\text{and }\overline{Y}\setminus Z(y_1(1)y_2(1)\cdots y_s(1))\rightarrow Y\text{ is an open embedding.}\end{array}\tag{16}$$

Lemma 6.2. *Suppose that (x,y) are prepared of type (s,r,l) and*

$$x_i=\prod_{j=1}^{r}x_j(1)^{a_{ij}}\text{ for }1\le i\le r$$

is a GMT in $x_1,\ldots,x_r$. Then there exists a SGMT

$$y_i=\prod_{j=1}^{s}y_j(1)^{b_{ij}}\text{ for }1\le i\le s$$

such that the variables $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1)\ge(s,r,l)$.

Proof. Let ν be the restriction of ν_e to the quotient field K of $\mathbb{C}[y_1,\ldots,y_s]$, which contains $\mathbb{C}[x_1,\ldots,x_r]$. The values $\nu(y_1),\ldots,\nu(y_s)$ are rationally independent and $\nu(x_1),\ldots,\nu(x_r)$ are rationally independent by Lemma 4.1. The inclusion $\mathbb{C}[x_1,\ldots,x_r]\rightarrow\mathbb{C}[y_1,\ldots,y_s]$ induces a dominant morphism $\mathbb{A}^s\rightarrow\mathbb{A}^r$ of nonsingular toric varieties. Let $\pi:Z\rightarrow\mathbb{A}^r$ be a projective morphism of nonsingular toric varieties such that $x_1(1),\ldots,x_r(1)$ are regular parameters in $\mathcal{O}_{Z,p}$ where p is the center of ν on Z. Let J be a monomial ideal in $\mathbb{C}[x_1,\ldots,x_r]$ whose blow up is Z. By principalization of ideals, there exists a projective morphism of toric varieties $\Lambda:W\rightarrow\mathbb{A}^s$ which is a product of blow ups of nonsingular subvarieties, such that $J\mathcal{O}_W$ is locally principal, so that the rational map $W\dashrightarrow Z$ is a morphism. Let q_1 be the center of ν on W. Since $\nu(y_1),\ldots,\nu(y_s)$ are rationally independent and Λ is toric, there exist regular parameters $\overline{y}_1,\ldots,\overline{y}_s$ in $\mathcal{O}_{W,q_1}$ and $b_{ij}\in\mathbb{N}$ with $\det(b_{ij})=\pm1$ such that

$$y_i=\prod_{j=1}^{s}\overline{y}_i^{b_{ij}}\text{ for }1\le i\le s.$$

W is the blow up of a (monomial) ideal H in $\mathbb{C}[y_1,\ldots,y_s]$. Let $Y_1\rightarrow\tilde{Y}$ be the blow up of H in a neighborhood of $e_{\tilde{Y}}$. Let e_{Y_1} be the center of e on Y_1. Then $\overline{y}_1,\ldots,\overline{y}_s,y_{s+1},\ldots,y_n$ are regular parameters in $\mathcal{O}^{\mathrm{an}}_{Y_1,e_{Y_1}}$, giving the conclusions of the lemma. ☐

Lemma 6.3. *Suppose that (x,y) are prepared of type (s,r,l), $1\le\overline{m}\le l$ and*

$$x_i=\prod_{j=1}^{r}x_j(1)^{a_{ij}}\text{ for }1\le i\le r$$

and

$$x_{r+\overline{m}}=\prod_{j=1}^{r}x_j(1)^{a_{r+\overline{m},j}}(x_{r+\overline{m}}(1)+\alpha)$$

with $0\ne\alpha\in\mathbb{C}$ is a GMT. Then there exists a SGMT

$$y_i=\prod_{j=1}^{s}y_j(1)^{b_{ij}}\text{ for }1\le i\le s$$

and

$$y_{s+\overline{m}}=\prod_{j=1}^{s}y_j(1)^{b_{s+\overline{m},j}}(y_{s+\overline{m}}(1)+\alpha)$$

such that the variables $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1)\ge(s,r,l)$.

Proof. Let $\overline{x}_1,\ldots,\overline{x}_n$ be the variables defined by (11) which lead to the variables $x_1(1),\ldots,x_n(1)$ of the statement of Lemma 6.3 by the analytic change of variables defined in Lemma 5.7.

Let ν be the restriction of ν_e to the quotient field K of $\mathbb{C}[y_1,\ldots,y_s,y_{s+\overline{m}}]$, which contains $\mathbb{C}[x_1,\ldots,x_r,x_{r+\overline{m}}]$. Then $\nu(y_1),\ldots,\nu(y_s)$ are rationally independent by Lemma 4.1 and $\nu(y_{s+\overline{m}})=\nu(x_{r+\overline{m}})$ is rationally dependent on $\nu(x_1),\ldots,\nu(x_r)$, hence $\nu(y_{s+\overline{m}})$ is rationally dependent on $\nu(y_1),\ldots,\nu(y_s)$. Let $\pi:Z\rightarrow\mathbb{A}^{r+1}$ be a projective morphism of nonsingular toric varieties such that $\overline{x}_1,\ldots,\overline{x}_r,\overline{x}_{r+\overline{m}}$ are regular parameters in $\mathcal{O}_{Z,p}$ where p is the center of ν on Z. We have that

$$\begin{array}{lll} x_i & = & \prod_{j=1}^{r}\overline{x}_j^{a_{ij}}(\overline{x}_{r+\overline{m}}+\overline{\alpha})^{a_{i,r+1}}\text{ for }1\le i\le r\text{ and}\\ x_{r+\overline{m}} & = & \prod_{j=1}^{r}\overline{x}_j^{a_{r+1,j}}(\overline{x}_{r+\overline{m}}+\overline{\alpha})^{a_{r+1,r+1}}\end{array}\tag{17}$$

where $0\ne\overline{\alpha}\in\mathbb{C}$.

Let J be a (monomial) ideal in $\mathbb{C}[x_1, \ldots, x_r, x_{r+\overline{m}}]$ whose blow up is Z. By principalization of ideals, there exists a toric projective morphism $\Lambda : W \to \mathbb{A}^{s+1}$ which is a product of blow ups of non singular varieties such that $J\mathcal{O}_W$ is locally principal. Let q_1 be the center of ν on W. Since $\nu(y_1), \ldots, \nu(y_s)$ are rationally independent, and Λ factors through Z, we have that $\mathcal{O}_{W,q_1}$ dominates $\mathcal{O}_{Z,p}$ and $\mathcal{O}_{W,q_1}$ has regular parameters $\overline{y}_1, \ldots, \overline{y}_s, \overline{y}_{s+\overline{m}}$ such that

$$(18)\qquad \begin{array}{lll} y_i & = & \prod_{j=1}^s \overline{y}_j^{b_{ij}} (\overline{y}_{s+\overline{m}} + \overline{\beta})^{b_{i,s+1}} \text{ for } 1 \le i \le s \text{ and} \\ y_{s+\overline{m}} & = & \prod_{j=1}^s \overline{y}_i^{b_{s+1,j}} (\overline{y}_{s+\overline{m}} + \overline{\beta})^{b_{s+1,s+1}} \end{array}$$

where $0 \neq \overline{\beta} \in \mathbb{C}$, $b_{ij} \in \mathbb{N}$ and $\mathrm{Det}(b_{ij}) = \pm 1$.

The variety W is the blow up of a monomial ideal H in $\mathbb{C}[y_1, \ldots, y_s, y_{s+\overline{m}}]$. Let $Y_1 \to \tilde{Y}$ be the blow up of H in a neighborhood of $e_{\tilde{Y}}$. Let e_{Y_1} be the center of e on Y_1. Then

$$\overline{y}_1, \ldots, \overline{y}_s, y_{s+1}, \ldots, y_{s+\overline{m}-1}, \overline{y}_{s+\overline{m}}, y_{s+\overline{m}+1}, \ldots, y_n$$

are regular parameters in $\mathcal{O}^{\mathrm{an}}_{Y_1,e_{Y_1}}$.

In $\mathcal{O}^{\mathrm{an}}_{X_1,e_{X_1}}$, we have the following relations between the variables $\overline{x}$ and $x(1)$.

$$(19)\qquad \begin{array}{lll} \overline{x}_i & = & (x_{r+\overline{m}}(1) + \alpha)^{\overline{c}\gamma_i} x_i(1) \text{ for } 1 \le i \le r \text{ and} \\ \overline{x}_{r+\overline{m}} & = & (x_{r+\overline{m}}(1) + \alpha)^{\overline{c}} - \overline{\alpha} \end{array}$$

with $\alpha^{\overline{c}} = \overline{\alpha}$ and

$$(a_{ij}) \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{\overline{c}} \end{pmatrix}$$

with

$$\overline{c} = \det \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ & \vdots & \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \det \begin{pmatrix} a_{11} & \cdots & a_{1,r+1} \\ & \vdots & \\ a_{r+1,1} & \cdots & a_{r+1,r+1} \end{pmatrix}.$$

In $\mathcal{O}^{\mathrm{an}}_{Y_1,e_{Y_1}}$, we have the following relations between the variables $\overline{y}$ and $y(1)$ of the proof of Lemma 5.7.

$$(20)\qquad \begin{array}{lll} \overline{y}_i & = & (y_{s+\overline{m}}(1) + \beta)^{\overline{d}\tau_i} y_i(1) \text{ for } 1 \le i \le s \text{ and} \\ \overline{y}_{s+\overline{m}} & = & (y_{s+\overline{m}}(1) + \beta)^{\overline{d}} - \overline{\beta} \end{array}$$

with $\beta^{\overline{d}} = \overline{\beta}$ and

$$(b_{ij}) \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_s \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{\overline{d}} \end{pmatrix}$$

with

$$\overline{d} = \det \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ & \vdots & \\ b_{s1} & \cdots & b_{ss} \end{pmatrix} \det \begin{pmatrix} b_{11} & \cdots & b_{1,s+1} \\ & \vdots & \\ b_{s+1,1} & \cdots & b_{s+1,s+1} \end{pmatrix}.$$

We have expressions

$$\overline{x}_i = x_1^{g_{i1}} \cdots x_r^{g_{ir}} x_{r+\overline{m}}^{g_{i,r+1}} \text{ for } 1 \le i \le r$$

and

$$\overline{x}_{r+\overline{m}}+\overline{\alpha}=x_1^{g_{r+1,1}}\cdots x_r^{g_{r+1,r}}x_{r+\overline{m}}^{g_{r+1,r+1}}$$

where $(g_{ij})=(a_{ij})^{-1}$ and

$$\overline{y}_i=y_1^{h_{i1}}\cdots y_s^{h_{is}}y_{s+\overline{m}}^{h_{i,s+1}}\text{ for }1\le i\le s$$

and

$$\overline{y}_{s+\overline{m}}+\overline{\beta}=y_1^{h_{s+1,1}}\cdots y_s^{h_{s+1,s}}y_{s+\overline{m}}^{h_{s+1,s+1}}$$

where $(h_{ij})=(b_{ij})^{-1}$.

Substituting (14), we have

$$\overline{x}_i=y_1^{d_{i1}}\cdots y_s^{d_{is}}y_{s+\overline{m}}^{d_{i,s+1}}\text{ for }1\le i\le r$$

and

$$\overline{x}_{r+\overline{m}}+\overline{\alpha}=y_1^{d_{r+1,1}}\cdots y_s^{d_{r+1,s}}y_{s+\overline{m}}^{d_{r+1,s+1}}$$

where

$$(d_{ik})=(a_{ij})^{-1}\begin{pmatrix}(c_{jk}) & 0\\ 0 & 1\end{pmatrix}.$$

We have

$$\begin{array}{lcl}\overline{x}_i & = & \overline{y}_1^{e_{i1}}\cdots\overline{y}_s^{e_{is}}(\overline{y}_{s+\overline{m}}+\overline{\beta})^{e_{i,s+1}}\text{ for }1\le i\le r\text{ and}\\ \overline{x}_{r+\overline{m}}+\overline{\alpha} & = & \overline{y}_1^{e_{r+1,1}}\cdots\overline{y}_s^{e_{r+1,s}}(\overline{y}_{s+\overline{m}}+\overline{\beta})^{e_{r+1,s+1}}\end{array}\tag{21}$$

where $(e_{ij})=(d_{ij})(h_{ij})^{-1}$. Since $\nu(\overline{x}_{r+\overline{m}}+\overline{\alpha})=\nu(\overline{y}_{s+\overline{m}}+\overline{\beta})=0$ and $\nu(\overline{y}_1),\ldots,\nu(\overline{y}_s)$ are rationally independent we have that

$$0=e_{r+1,1}=\cdots=e_{r+1,s}.$$

We then have that $e_{s+1,s+1}\neq 0$ since $\operatorname{rank}(e_{ij})=r+1$. We have that $e_{ij}\ge 0$ for $1\le i\le r+1$ and $1\le j\le s+1$ since Λ factors through Z. We compute

$$\begin{array}{rcl}(e_{ij})\begin{pmatrix}\tau_1\\ \vdots\\ \tau_s\\ 1\end{pmatrix} & = & (a_{ij})^{-1}\begin{pmatrix}(c_{ij}) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \vdots\\ 0\\ \frac{1}{\overline{d}}\end{pmatrix}\\ & = & (a_{ij})^{-1}\begin{pmatrix}0\\ \vdots\\ 0\\ \frac{1}{\overline{d}}\end{pmatrix}=\frac{\overline{c}}{\overline{d}}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_r\\ 1\end{pmatrix}.\end{array}$$

Substituting (19) and (20) into (21), we obtain

$$x_i(1)(x_{r+\overline{m}}(1)+\alpha)^{\overline{c}\gamma_i}=y_1(1)^{e_{i1}}\cdots y_s(1)^{e_{s1}}(y_{s+\overline{m}}(1)+\beta)^{\overline{c}\gamma_i}\text{ for }1\le i\le r$$

and

$$(\overline{x}_{r+\overline{m}}(1)+\alpha)^{\overline{c}}=(\overline{y}_{s+\overline{m}}(1)+\beta)^{\overline{c}}.$$

We thus have an expression (after possibly replacing $\overline{y}_{s+\overline{m}}$ with its product times a root of unity)

$$x_i(1)=\prod_{j=1}^{s}y_j(1)^{e_{ij}}\text{ for }1\le i\le r$$

and

$$x_{r+\overline{m}}(1)=y_{s+\overline{m}}(1)$$

giving the conclusions of the lemma. $\square$

Lemma 6.4. *Suppose that (x,y) are prepared of type (s,r,l), $\overline{m} > l$ and we have an expression*

$$x_{r+\overline{m}} = y_1^{c_{r+1,1}} \cdots y_s^{c_{r+1,s}} u$$

where $u \in \mathbb{C}\{\{y_1,\ldots,y_n\}\}$ is a unit and

$$x_i = \prod_{j=1}^{r} x_j(1)^{a_{ij}} \text{ for } 1 \le i \le r \text{ and}$$

$$x_{r+\overline{m}} = \prod_{j=1}^{r} x_j(1)^{a_j}(x_{r+\overline{m}}(1)+\alpha) \text{ with } 0 \ne \alpha \in \mathbb{C}$$

is a GMT in $x_1,\ldots,x_r,x_{r+\overline{m}}$. Then there exists a SGMT

$$y_i = \prod_{j=1}^{s} y_j(1)^{b_{ij}} \text{ for } 1 \le i \le s$$

in $y_1,\ldots,y_s$ such that the variables $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1) \ge (s,r,l)$.

Proof. By Lemma 5.7, the GMT $(x) \to (x(1))$ is determined by a monoidal transform

$$x_i = \left(\prod_{j=1}^{r} \overline{x}_i^{g_{ij}}\right)(\overline{x}_{r+\overline{m}}+\overline{\alpha})^{g_{i,r+1}} \text{ for } 1 \le i \le r \text{ and}$$

$$x_{r+\overline{m}} = \left(\prod_{j=1}^{r} \overline{x}_i^{g_{r+1,j}}\right)(\overline{x}_{r+\overline{m}}+\overline{\alpha})^{g_{r+1,r+1}}$$

where $\det(g_{ij}) = \pm 1$ and

$$(22)\qquad \begin{array}{lcl} x_i(1) & = & (\overline{x}_{r+\overline{m}}+\overline{\alpha})^{\lambda_i}\overline{x}_i \text{ for } 1 \le i \le r \text{ and} \\ x_{r+\overline{m}}(1) & = & (\overline{x}_{r+\overline{m}}+\overline{\alpha})^{\lambda} - \overline{\alpha}^{\lambda},\ \alpha = \overline{\alpha}^{\lambda} \end{array}$$

for suitable $\lambda_i, \lambda \in \mathbb{Q}$ (with $\lambda \ne 0$). Letting $(e_{ij}) = (g_{ij})^{-1}$ and $(d_{ij}) = (g_{ik})^{-1}(c_{kj})$, we have

$$\overline{x}_i = \left(\prod_{j=1}^{s} y_j^{d_{ij}}\right) u^{e_{i,r+1}} \text{ for } 1 \le i \le r \text{ and}$$

$$\overline{x}_{r+\overline{m}} + \overline{\alpha} = \left(\prod_{j=1}^{s} y_j^{d_{r+1,j}}\right) u^{e_{r+1,r+1}}.$$

The values $\nu_e(y_1),\ldots,\nu_e(y_s)$ are rationally independent by Lemma 4.1. Since

$$\nu_e(x_{r+\overline{m}}+\overline{\alpha}) = \nu_e(u) = 0,$$

we have that $d_{r+1,j} = 0$ for $1 \le j \le s$. Thus by (22),

$$(23)\qquad x_{r+\overline{m}}(1) = u^{\lambda e_{r+1,r+1}} - \alpha \in \mathbb{C}\{\{y_1,\ldots,y_n\}\}.$$

Write

$$\prod_{j=1}^{s} y_j^{d_{ij}} = \frac{M_i}{N_i}$$

where M_i, N_i are monomials in $y_1, \ldots, y_s$ for $1 \leq i \leq r$. Let K be the ideal $K = \prod_{i=1}^{s}(M_i, N_i)$ in $\mathbb{C}\{\{y_1, \ldots, y_s\}\}$. By Lemma 5.5, there exists a (monomial) SGMT in $y_1, \ldots, y_s$

$$y_i = \prod_{j=1}^{s} y_j(1)^{b_{ij}} \text{ for } 1 \leq i \leq s$$

such that $K\mathcal{O}^{\mathrm{an}}_{Y(1),e_{Y(1)}}$ is a principal ideal. $y_1(1), \ldots, y_s(1)$ are independent by Lemma 5.2. Since $\nu_e(M_i/N_i) = \nu_e(x_i) > 0$ we have that N_i divides M_i in $\mathcal{O}^{\mathrm{an}}_{Y(1),e_{Y(1)}}$ for $1 \leq i \leq s$ and so we have an expression

$$\overline{x}_i = \left(\prod_{j=1}^{s} y_j(1)^{c_{ij}(1)}\right) u^{e_{i,r+1}} \text{ for } 1 \leq i \leq r$$

with $c_{ij}(1) \in \mathbb{N}$. Since $x_i(1)$ is necessarily a Laurent monomial in $y_1(1), \ldots, y_s(1)$ for $1 \leq i \leq s$, comparing with (22), we see that

$$x_i(1) = \prod_{j=1}^{s} y_j(1)^{c_{ij}(1)} \text{ for } 1 \leq i \leq r.$$

Since $x_{r+\overline{m}}(1) \in \mathbb{C}\{\{y_1(1), \ldots, y_n(1)\}\}$ by (23), we have attained the conclusions of the lemma.

$\square$

Suppose that (x, y) are prepared of type (s, r, l). We will perform sequences of transformations of the following 10 types for $1 \leq i \leq 10$ each of which will be called a transformation of type i) from the variables (x, y) to $(x(1), y(1))$. The variables $x(1)$ and $y(1)$ are respective regular parameters in $\mathcal{O}^{\mathrm{an}}_{X(1),e_{X(1)}}$ and $\mathcal{O}^{\mathrm{an}}_{Y(1),e_{Y(1)}}$ from the corresponding diagram of quasi regular analytic maps

$$\begin{array}{ccc} Y(1) & \stackrel{\varphi(1)}{\rightarrow} & X(1) \\ \downarrow & & \downarrow \\ \tilde{Y} & \stackrel{\tilde{\varphi}}{\rightarrow} & \tilde{X} \end{array}$$

where $Y(1) \to \tilde{Y} \to Y \in e$ and $X(1) \to \tilde{X} \to X \in e$. We have that $(x(1), y(1))$ is prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) \geq (s, r, l)$ for all 10 types of transformations. The fact that none of s, r or $r + l$ can go down after a transformation follows from Lemmas 5.2, 5.7 and 5.6. Existence of transformations of types 2) and 4) follow from Lemmas 6.2 and 6.3. A transformation of type 9) will be constructed in the proof of Proposition 8.9 (using Lemma 6.4).

Transformations of types 1) to 4) are the most basic and are used most of the time. Transformations of types 1) - 6) and 1) - 8) are used in blocks, depending on the lemma or proposition. Transformations of type 3), 5) or 10) are often used to make a Tschirnhaus transformation (Lemma 5.8). A transformation of type 8) is often used to make a change of variables, giving an increase in r. A transformation of type 9) is used at the end of the proof of Proposition 8.9.

1) A (necessarily monomial) SGMT in $y_1, \ldots, y_s$,

$$y_i = \prod_{j=1}^{s} y_j(1)^{b_{ij}} \text{ for } 1 \leq i \leq s,$$

with $\mathrm{Det}(b_{ij}) = \pm 1$.

2) A (necessarily monomial) SGMT in $x_1, \ldots, x_r$ followed by a (necessarily monomial) SGMT in $y_1, \ldots, y_s$,

$$x_i = \prod_{j=1}^{r} x_j(1)^{a_{ij}} \text{ for } 1 \le i \le r$$

and

$$y_i = \prod_{j=1}^{s} y_j(1)^{b_{ij}} \text{ for } 1 \le i \le s$$

with $\mathrm{Det}(a_{ij}) = \pm 1$ and $\mathrm{Det}(b_{ij}) = \pm 1$.

3) A change of variables $x_{r+\overline{m}}(1) = x_{r+\overline{m}} - \Phi$ for some $\overline{m}$ with $1 \le \overline{m} \le l$ and $\Phi \in \mathbb{C}\{\{x_1, \ldots, x_{r+\overline{m}-1}\}\}$, followed by a change of variables $y_{s+\overline{m}}(1) = y_{s+\overline{m}} - \Phi$.

4) A SGMT in $x_1, \ldots, x_r, x_{r+\overline{m}}$ followed by a SGMT in $y_1, \ldots, y_s, y_{s+\overline{m}}$ for some $\overline{m}$ with $1 \le \overline{m} \le l$,

$$x_i = \prod_{j=1}^{r} x_j(1)^{a_{ij}} \text{ for } 1 \le i \le r \text{ and } x_{r+\overline{m}} = \prod_{j=1}^{r} x_j(1)^{a_j}(x_{r+\overline{m}}(1) + \alpha)$$

for some $0 \ne \alpha \in \mathbb{C}$, and

$$y_i = \prod_{j=1}^{s} y_j(1)^{b_{ij}} \text{ for } 1 \le i \le s \text{ and } y_{s+\overline{m}} = \prod_{j=1}^{s} y_j(1)^{b_j}(y_{s+\overline{m}}(1) + \alpha)$$

with $\mathrm{Det}(a_{ij}) \ne 0$ and $\mathrm{Det}(b_{ij}) \ne 0$ and $\prod_{j=1}^{s} y_j(1)^{b_j} = \prod_{j=1}^{r} x_j(1)^{a_j}$.

5) A change of variables $y_{s+\overline{m}}(1) = F$ with $F \in \mathbb{C}\{\{y_1, \ldots, y_{s+\overline{m}}\}\}$ and

$$\mathrm{ord}\ F(0, \ldots, 0, y_{s+\overline{m}}) = 1$$

for some $\overline{m}$ with $\overline{m} > l$.

6) A SGMT in $y_1, \ldots, y_s, y_{s+\overline{m}}$, for some $\overline{m}$ with $l + 1 \le \overline{m} \le n - s$.

7) An interchange of variables y_{s+i} and $y_{s+\overline{m}}$ with $s + l < s + i < s + \overline{m} \le n$.

8) A change of variables, replacing y_i with $y_i \gamma^{c_i}$ for $1 \le i \le s$ for some unit $\gamma \in \mathbb{C}\{\{y_1, \ldots, y_n\}\}$ and $c_i \in \mathbb{Q}$ such that the form (14) is preserved.

9) A SGMT in $x_1, \ldots, x_r, x_{r+\overline{m}}$ followed by a SGMT in $y_1, \ldots, y_s$ (supposing that $\overline{m} > l$ and

$$x_{r+\overline{m}} = y_1^{b_1} \cdots y_s^{b_s} u$$

where $u \in \mathbb{C}\{\{y_1, \ldots, y_n\}\}$ is a unit),

$$x_i = \prod_{j=1}^{r} x_j(1)^{a_{ij}} \text{ for } 1 \le i \le r \text{ and } x_{r+\overline{m}} = \prod_{j=1}^{r} x_j(1)^{a_j}(x_{r+\overline{m}}(1) + \alpha)$$

for some $0 \ne \alpha \in \mathbb{C}$, and

$$y_i = \prod_{j=1}^{s} y_j(1)^{b_{ij}} \text{ for } 1 \le i \le s$$

with $\mathrm{Det}(b_{ij}) = \pm 1$ and $\mathrm{Det}(a_{ij}) \ne 0$ and $\prod_{j=1}^{s} y_j^{b_j} = \prod_{j=1}^{r} x_j(1)^{a_j}$.

10) A change of variables, replacing $x_{r+\overline{m}}$ with $x_{r+\overline{m}} - \Phi$ for some $l < \overline{m} \le m - r$ and $\Phi \in \mathbb{C}\{\{x_1, \ldots, x_{r+\overline{m}-1}\}\}$.

In the following, we will assume that (s, r, l) is preserved by these transformations. If this does not hold, then we just start over again with the assumption of the higher (s, r, l). As these numbers cannot increase indefinitely, we will eventually reach a situation where they remain stable under the above transformations.

A sequence of transformations

$$(x, y) \to (x(1), y(1)) \to \cdots \to (x(t-1), y(t-1)) \to (x(t), y(t))$$

will be called a sequence of transformations from (x, y) to $(x(t), y(t))$.

Observe that a sequence of transformations (which are of types 1) - 10)) satisfy the condition (16).

7. A decomposition of series

In this section, suppose that (x, y) are prepared of type (s, r, l). As commented after (14), we have a natural inclusion of formal power series rings

$$\mathbb{C}[[x_1, \ldots, x_{r+l}]] \subset \mathbb{C}[[y_1, \ldots, y_{s+l}]].$$

Definition 7.1. *Suppose that $g \in k[[y_1, \ldots, y_n]]$. We will say that g is algebraic over $x_1, \ldots, x_{r+l}$ if $g \in \mathbb{C}[[y_1, \ldots, y_{s+l}]]$ and g has an expansion*

$$g = \sum a_{i_1,\ldots,i_{s+l}} y_1^{i_1} \cdots y_s^{i_s} y_{s+1}^{i_{s+1}} \cdots y_{s+l}^{i_{s+l}} \tag{24}$$

where $a_{i_1,\ldots,i_{s+l}} \in \mathbb{C}$ is nonzero only if

$$\text{rank} \begin{pmatrix} c_{11} & \cdots & c_{1s} \\ & \vdots & \\ c_{r1} & \cdots & c_{rs} \\ i_1 & \cdots & i_s \end{pmatrix} = r.$$

Observe that the property that g is algebraic over $x_1, \ldots, x_{r+l}$ is preserved by a transformation of type 8).

Lemma 7.2. *Suppose that $x_1^{b_1} \cdots x_r^{b_r}$ with $b_1, \ldots, b_r \in \mathbb{Z}$ is such that $\prod_{i=1}^r (y_1^{c_{i1}} \cdots y_s^{c_{is}})^{b_i} \in \mathbb{C}[y_1, \ldots, y_s]$ is algebraic over $x_1, \ldots, x_r$. Then there exists a SGMT*

$$x_i = \prod_{j=1}^r x_j(1)^{a_{ij}} \text{ for } 1 \le i \le r$$

such that

$$x_1^{b_1} \cdots x_r^{b_r} = x_1(1)^{b_1(1)} \cdots x_r(1)^{b_r(1)}$$

with $b_i(1) \in \mathbb{N}$ for all i.

Proof. Let ν be the restriction of ν_e to the quotient field of $\mathbb{C}[y_1, \ldots, y_s]$. We have $\nu(x_1^{b_1} \cdots x_r^{b_r}) \ge 0$. Write $x_1^{b_1} \cdots x_r^{b_r} = \frac{M_1}{M_2}$ where M_1 and M_2 are monomials in $x_1, \ldots, x_r$. We have that $\nu(M_1) \ge \nu(M_2)$. By Lemma 5.5, there exists a monomial SGMT in $x_1, \ldots, x_r$ such that the ideal generated by M_1 and M_2 in $\mathcal{O}^{\text{an}}_{X(1),e_{X(1)}}$ is principal. Since $\nu(M_1) \ge \nu(M_2)$, we have that M_2 divides M_1 in $\mathcal{O}^{\text{an}}_{X(1),e_{X(1)}}$, giving the conclusions of the lemma. $\square$

Suppose that $g \in \mathbb{C}[[y_1, \ldots, y_{s+l}]]$. As on page 1540 of [19], we have an expression

$$g = \sum_{[\Lambda] \in (\mathbb{Z}^s/(\mathbb{Q}^r C) \cap \mathbb{Z}^s)} h_{[\Lambda]} \tag{25}$$

where

$$h_{[\Lambda]} = \sum_{\alpha \in \mathbb{N}^s \mid [\alpha] = [\Lambda]} g_\alpha y_1^{\alpha_1} \cdots y_s^{\alpha_s} \tag{26}$$

with $g_\alpha \in \mathbb{C}[[y_{s+1}, \ldots, y_{s+l}]]$.

If $g \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$ then each $h_{[\Lambda]} \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$ by the criterion of (3).

Proposition 7.3. *Suppose that $\Lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{N}^s$ is fixed. Then there exists a SGMT of type 2), $(x, y) \mapsto (x(1), y(1))$, $w_1, \ldots, w_r \in \mathbb{N}$ and $d \in \mathbb{Z}_{>0}$ such that*

$$\delta_{[\Lambda]} := \frac{h_{[\Lambda]}}{y_1^{\lambda_1} \cdots y_s^{\lambda_s}} x_1^{w_1} \cdots x_r^{w_r} \in \mathbb{C}[[x_1(1)^{\frac{1}{d}}, \ldots, x_r(1)^{\frac{1}{d}}, x_{r+1}(1), \ldots, x_{r+l}(1)]]. \tag{27}$$

If $[\Lambda] = 0$, we further have

$$h_{[\Lambda]} \in \mathbb{C}[[x_1(1)^{\frac{1}{d}}, \ldots, x_r(1)^{\frac{1}{d}}, x_{r+1}(1), \ldots, x_{r+l}(1)]].$$

If $g \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$, then $\delta_{[\Lambda]} \in \mathbb{C}\{\{x_1(1)^{\frac{1}{d}}, \ldots, x_r(1)^{\frac{1}{d}}, x_{r+1}(1), \ldots, x_{r+l}(1)\}\}$ by the criterion (3).

Proof. Write $C = (C_1, \ldots, C_s)$ and let $\Phi : \mathbb{Q}^r \to \mathbb{Q}^s$ be defined by $\Phi(v) = vC$ for $v \in \mathbb{Q}^r$. Φ is injective since C has rank r. Let $G = \Phi^{-1}(\mathbb{Z}^s)$. For $\Lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{N}^s$, define

$$P_\Lambda = \{v \in \mathbb{Q}^r \mid vC_i + \lambda_i \geq 0 \text{ for } 1 \leq i \leq s\}.$$

For $\Lambda \in \mathbb{N}^s$, we have

$$h_{[\Lambda]} = y_1^{\lambda_1} \cdots y_s^{\lambda_s} \left(\sum_{v = (v_1, \ldots, v_r) \in G \cap P_\Lambda} x_1^{v_1} \cdots x_r^{v_r} g_v \right)$$

where $g_v \in \mathbb{C}[[x_{r+1}, \ldots, x_{r+l}]]$ and we have reindexed the $g_\alpha = g_{vC+\Lambda}$ in (26) as g_v. Let

$$H = \{v \in \mathbb{Z}^r \mid vC_i \geq 0 \text{ for } 1 \leq i \leq s\},$$

$$I = \{v \in G \mid vC_i \geq 0 \text{ for } 1 \leq i \leq s\}$$

and for $\Lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{N}^s$,

$$M_\Lambda = \{v \in G \mid vC_i + \lambda_i \geq 0 \text{ for } 1 \leq i \leq s\}.$$

We have that P_Λ is a rational polyhedral set in $\mathbb{Q}^r$ whose associated cone is

$$\sigma = \{v \in \mathbb{Q}^r \mid vC_i = 0 \text{ for } 1 \leq i \leq s\} = \{0\}.$$

Let $W = \mathbb{Q}^r$. We have that G is a lattice in W and P_Λ is strongly convex. Thus $M_\Lambda = P_\Lambda \cap G$ is a finitely generated module over the semigroup I (cf. Theorem 7.1 [29]). Let $\overline{n} = [G : \mathbb{Z}^r]$. We have that $\overline{n}x \in H$ for all $x \in I$. Gordan's Lemma (cf. Proposition 1, page 12 [34]) implies that H and I are finitely generated semigroups. There exist $w_1, \ldots, w_{\overline{l}} \in I$ which generate I as a semigroup and there exist $\overline{v}_1, \ldots, \overline{v}_{\overline{a}} \in H$ which generated H as a semigroup. Then the finite set

$$\{a_1 w_1 + \cdots a_{\overline{l}} w_{\overline{l}} \mid a_i \in \mathbb{N} \text{ and } 0 \leq a_i \leq \overline{n} \text{ for } 1 \leq i \leq \overline{l}\}$$

generates I as an H-module. We thus have that M_Λ is a finitely generated module over the semigroup H. Thus there exist $\overline{u}_1,\ldots,\overline{u}_{\overline{b}}\in M_\Lambda$ such that if $v=(v_1,\ldots,v_r)\in M_\Lambda$, then

$$v=\overline{u}_i+\sum_{j=1}^{\overline{a}}n_j\overline{v}_j$$

for some $1\le i\le b$ and $n_1,\ldots,n_{\overline{a}}\in\mathbb{N}$. Thus

$$x_1^{v_1}\cdots x_r^{v_r}=x_1^{\overline{u}_{i,1}}\cdots x_r^{\overline{u}_{i,r}}\prod_{j=1}^{\overline{a}}(x_1^{\overline{v}_{j,1}}\cdots x_r^{\overline{v}_{j,r}})^{n_j}$$

where $\overline{u}_i=(\overline{u}_{i,1},\ldots,\overline{u}_{i,r})$ for $1\le i\le\overline{b}$ and $\overline{v}_j=(\overline{v}_{j,1},\ldots,\overline{v}_{j,r})$ for $1\le j\le\overline{a}$. By Lemma 7.2 and Lemma 6.2, there exists a transformation of type 2) such that for $1\le j\le\overline{a}$,

$$x_1^{\overline{v}_{j,1}}\cdots\overline{x}_r^{\overline{v}_{j,r}}=x_1(1)^{\overline{v}(1)_{j,1}}\cdots x_r(1)^{\overline{v}(1)_{j,r}}$$

with $(\overline{v}(1)_{j,1},\ldots,\overline{v}(1)_{j,r})\in\mathbb{N}^r$ for $1\le j\le\overline{a}$. We then have expressions of all $\Lambda=(\lambda_1,\ldots,\lambda_s)\in\mathbb{N}^s$, where $\overline{u}_1,\ldots,\overline{u}_{\overline{b}}\in\mathbb{Q}^r$ depend only on Λ,

$$h_{[\Lambda]}=y_1(1)^{\lambda_1(1)}\cdots y_s(1)^{\lambda_s(1)}\left[\sum_{i=1}^{\overline{b}}x_1(1)^{\overline{u}_{i,1}(1)}\cdots x_r(1)^{\overline{u}_{i,r}(1)}g_i\right]$$

where $g_i\in\mathbb{C}[[x_{r+1}(1),\ldots,x_{r+l}(1)]]$,

$$\Lambda(1):=(\lambda_1(1),\ldots,\lambda_s(1))=\Lambda(b_{ij})$$

and

$$\overline{u}(1)_i=(\overline{u}_{i,1}(1),\ldots,\overline{u}_{i,r}(1))=\overline{u}_i(a_{ij}).$$

If $\Lambda=0$, we have $M_\Lambda=I$ so that $x_1^{\overline{u}_{i,1}}\cdots\overline{x}_r^{\overline{u}_{i,r}}$ is a monomial in $y_1,\ldots,y_s$ for $1\le i\le\overline{b}$, so we can construct a transformation of type 2), $(x,y)\mapsto(x(1),y(1))$ so that we also have that the $\overline{u}_i(1)$ satisfy $\overline{u}_i(1)\in\mathbb{Q}^r_{\ge0}$ for $1\le i\le\overline{b}$.

Now let d be a common denominator of the coefficients of the $\overline{u}_i(1)$ for $1\le i\le\overline{b}$. If $[\Lambda]=0$, we have that

$$h_{[\Lambda]}\in\mathbb{C}[[x_1(1)^{\frac{1}{d}},\ldots,x_r(1)^{\frac{1}{d}},x_{r+1}(1),\ldots,x_{r+l}(1)]].$$

If $[\Lambda]\ne0$, we choose $w=(w_1,\ldots,w_r)\in\mathbb{N}^r$ such that $w+\overline{u}_i\in\mathbb{Q}^r_{\ge0}$ for $1\le i\le\overline{b}$. Then

$$\frac{h_{[\Lambda]}}{y_1^{\lambda_1}\cdots y_s^{\lambda_s}}x_1^{w_1}\cdots x_r^{w_r}\in\mathbb{C}[[x_1(1)^{\frac{1}{d}},\ldots,x_r(1)^{\frac{1}{d}},x_{r+1}(1),\ldots,x_{r+l}(1)]].$$

$\square$

Lemma 7.4. *Suppose that $f\in\mathbb{C}\{\{x_1,\ldots,x_m\}\}\subset\mathbb{C}[[y_1,\ldots,y_n]]$ is algebraic over $x_1,\ldots,x_{r+l}$. Then $f\in\mathbb{C}\{\{x_1,\ldots,x_{r+l}\}\}$.*

Proof. By Proposition 7.3 and by the criterion of (3), there exists a monomial GMT

$$\begin{array}{lcl} x_1 &=& x_1(1)^{a_{11}(1)}\cdots x_r(1)^{a_{1r}(1)}\\ &\vdots&\\ x_r &=& x_1(1)^{a_{r1}(1)}\cdots x_r(1)^{a_{rr}(1)}\end{array}\tag{28}$$

with $\mathrm{Det}(a_{ij}(1)) = \pm 1$ and $d \in \mathbb{Z}_+$ such that

$$f \in \mathbb{C}\{\{x_1(1)^{\frac{1}{d}}, \ldots, x_r(1)^{\frac{1}{d}}, x_{r+1}, \ldots, x_{r+l}\}\}.$$

Let

$$\begin{array}{lll} g(z) & = & \prod_{i_1,\ldots,i_r=1}^{d}(z - f(\omega^{i_1}x_1(1)^{\frac{1}{d}}, \ldots, \omega^{i_r}x_r(1)^{\frac{1}{d}}, x_{r+1}, \ldots, x_{r+l})) \\ & & \in \mathbb{C}\{\{x_1(1), \ldots, x_r(1), x_{r+1}, \ldots, x_{r+l}\}\}[z] \end{array}$$

where ω is a primitive complex d-th root of unity. We have that f is integral over $\mathbb{C}\{\{x_1(1), \ldots, x_r(1), x_{r+1}, \ldots, x_{r+l}\}\}$ since f is a root of $g(z) = 0$. But

$$f \in \mathbb{C}\{\{x_1(1), \ldots, x_r(1), x_{r+1}, \ldots, x_m\}\}$$

and $\mathbb{C}\{\{x_1(1), \ldots, x_r(1), x_{r+1}, \ldots, x_{r+l}\}\}$ is integrally closed in

$$\mathbb{C}\{\{x_1(1), \ldots, x_r(1), x_{r+1}, \ldots, x_m\}\}$$

so $f \in \mathbb{C}\{\{x_1(1), \ldots, x_r(1), x_{r+1}, \ldots, x_{r+l}\}\}$. Substituting (28) into the series expansion of f in terms of $x_1, \ldots, x_m$ we obtain that $f \in \mathbb{C}\{\{x_1, \ldots, x_{r+l}\}\}$.

□

Lemma 7.5. *Suppose that $g \in \mathbb{C}[[y_1, \ldots, y_{s+l}]]$ has an expression $g = \sum h_{[\Lambda]}$ and one of the transformations 1) - 4) are performed. Then $g \in \mathbb{C}[[y_1(1), \ldots, y_{s+l}(1)]]$ and if $g = \sum h'_{[\Lambda']}$ is the decomposition in terms of the variables $y_1(1), \ldots, y_{s+l}(1)$ and $x_1(1), \ldots, x_{r+l}(1)$, then*

(29) $$h_{[\Lambda]} = h'_{[\Lambda\overline{B}]}$$

where

$$\overline{B} = \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ & \vdots & \\ b_{s1} & \cdots & b_{ss} \end{pmatrix}$$

with b_{ij} defined as in the definitions of types 1), 2) and 4) (and with $\overline{B}$ being the identity matrix for a transformation of type 3).

In particular, if a transformation of type 1) - 10) is performed, then $f \in \mathbb{C}[[y_1, \ldots, y_n]]$ is algebraic over $x_1, \ldots, x_{r+l}$ if and only if f is algebraic over $x_1(1), \ldots, x_{r+l}(1)$.

Proof. We will prove (29) in the case of a transformation of type 4). The other cases are simpler. With the notation of (26), we have expansions

$$g_\alpha = \sum_i (y_1(1)^{b_1} \cdots y_s(1)^{b_s})^i g_{\alpha,i}$$

with $g_{\alpha,i} \in \mathbb{C}[[y_{s+1}(1), \ldots, y_{s+l}(1)]]$ so

$$\begin{array}{lll} h_{[\Lambda]} & = & \sum_{[\alpha]=[\Lambda]} \prod_{j=1}^{s}(y_1(1)^{b_{j1}} \cdots y_s(1)^{b_{js}})^{\alpha_j} (\sum_i (y_1(1)^{b_1} \cdots y_s(1)^{b_s})^i g_{\alpha,i}) \\ & = & \sum_{\overline{\alpha}} y_1(1)^{\overline{\alpha}_1} \cdots y_s^{\overline{\alpha}_s} \left(\sum_i (y_1(1)^{b_1} \cdots y_s(1)^{b_s})^i g_{\alpha,i}\right) \end{array}$$

where

(30) $$\alpha\overline{B} = \overline{\alpha}$$

with $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_s)$. Write

$$\overline{A} = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ & \vdots & \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}$$

and

$$C(1)=\begin{pmatrix} c_{11}(1) & \cdots & c_{1s}(1)\\ & \vdots & \\ c_{r1}(1) & \cdots & c_{rs}(1)\end{pmatrix}.$$

We showed in the proof of Lemma 6.3 (where (e_{ij}) is defined) that

$$A(e_{ij})=\begin{pmatrix} C & 0\\ 0 & 1\end{pmatrix}B.$$

We have that

$$A=\begin{pmatrix} \overline{A} & *\\ * & *\end{pmatrix},\ B=\begin{pmatrix} \overline{B} & *\\ * & *\end{pmatrix},\ (e_{ij})=\begin{pmatrix} C(1) & *\\ 0 & *\end{pmatrix}.$$

We obtain that

$$\overline{A}C(1)=C\overline{B}. \tag{31}$$

From $y_1(1)^{b_1}\cdots y_s(1)^{b_s}=x_1(1)^{a_1}\cdots x_r(1)^{a_r}$ we obtain

$$(a_1,\ldots,a_r)C(1)=(b_1,\ldots,b_s)$$

and so

$$(b_1,\ldots,b_s)\in\mathbb{Q}^rC(1)\cap\mathbb{Z}^s.$$

Since $\overline{A}$ and $\overline{B}$ are invertible with integral coefficients, we have from (31) that for $\alpha,\beta\in\mathbb{Z}^s$, $\alpha-\beta\in\mathbb{Q}^rC\cap\mathbb{Z}^s$ if and only if $\alpha\overline{B}-\beta\overline{B}\in\mathbb{Q}^rC(1)\cap\mathbb{Z}^s$, from which we obtain (29). $\square$

8. Monomialization

Lemma 8.1. *Suppose that the variables (x,y) are prepared of type (s,r,l) and there exists t with $r<t\le r+l$ such that $x_1,\ldots,x_r,x_t$ are independent. Then there exists a transformation of type 6) with $\overline{m}=t-r$, possibly followed by a tranformation of type 8) $(x,y)\to(x(1),y(1))$ such that $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_2)>(s,r,l)$.*

Proof. Without loss of generality, we may assume that $t=r+1$. Since $y_1,\ldots,y_s$ are independent and $y_1,\ldots,y_s,y_{s+1}$ are dependent, there exists by Lemmas 5.7 and 5.4 a SGMT $(y)\to(y(1))$ (a transformation of type 6) with $\overline{m}=t-r=1$) defined by

$$y_i=\prod_{j=1}^s y_j(1)^{b_{ij}}\text{ for }1\le j\le s\text{ and}$$

$$y_{s+1}=\prod_{j=1}^s y_j(1)^{b_j}(y_{s+1}(1)+\alpha)\text{ with }\alpha\ne 0.$$

This gives us an expression

$$x_i=\prod_{j=1}^s y_j(1)^{c_{ij}(1)}\text{ for }1\le i\le r\text{ and}$$

$$x_{r+1}=\prod_{j=1}^s y_j(1)^{b_j}(y_{s+1}(1)+\alpha).$$

If $s_1 > s$ we are done. Otherwise, we must have that

$$\operatorname{rank}\begin{pmatrix} c_{11}(1) & \cdots & c_{1s}(1) \\ & \vdots & \\ c_{r1}(1) & \cdots & c_{rs}(1) \\ b_1 & \cdots & b_s \end{pmatrix} = r+1$$

since $x_1, \ldots, x_{r+1}$ are independent. Thus after making a change of variables in $y_1, \ldots, y_s$ (a transformation of type 8)) with $\gamma = (y_{s+1}(1) + \alpha))$ we obtain an increase $r_1 > r$ (and $(s_1, r_1, l_1) > (s, r, l)$). $\square$

Lemma 8.2. *Suppose that (x, y) are prepared of type (s, r, l) and $g \in \mathbb{C}\{\{x_1, \ldots, x_{r+l}\}\}$. Then either there exists a sequence of transformations $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) > (s, r, l)$ or there exists a sequence of transformations of the types 2) - 4) $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) = (s, r, l)$ and we have an expression*

$$g = x_1(1)^{d_1} \cdots x_r(1)^{d_r} u$$

with $u \in \mathbb{C}\{\{x_1(1), \ldots, x_{r+l}(1)\}\}$ a unit.

Proof. In the course of the proof, we may assume that all transformations do not lead to an increase in (s, r, l). We will establish the lemma by induction on t with $g \in \mathbb{C}\{\{x_1, \ldots, x_t\}\}$ for $r \le t \le r + l$. We will establish the lemma then with the further restriction that all transformations of types 3) and 4) have $\overline{m} \le t - r$ and we will obtain $u \in \mathbb{C}\{\{x_1(1), \ldots, x_t(1)\}\}$.

We first prove the lemma for $t = r$, so suppose $g \in \mathbb{C}\{\{x_1, \ldots, x_r\}\}$. Expand

$$g = \sum a_{i_1,\ldots,i_r} x_1^{i_1} \cdots x_r^{i_r} \text{ with } a_{i_1,\ldots,i_r} \in \mathbb{C}.$$

Let I be the ideal

$$I = (x_1^{i_1} \cdots x_r^{i_r} \mid a_{i_1,\ldots,i_r} \neq 0).$$

The ideal I is generated by $x_1^{i_1(1)} \cdots x_r^{i_r(1)}, \ldots, x_1^{i_1(k)} \cdots x_r^{i_r(k)}$ for some $i_1(1), \ldots, i_r(k)$ with $k \in \mathbb{Z}_{>0}$. By performing a transformation of type 2) $(x, y) \to (x(1), y(1))$ we may principalize the ideal I (by Lemma 5.5). Suppose that $x_1(1)^{a_1} \cdots x_r(1)^{a_r}$ is a generator of $I\mathcal{O}^{\mathrm{an}}_{X(1), e_{X(1)}}$. Then since $x_1, \ldots, x_r$ are independent, we have that $g = x_1(1)^{a_1} \cdots x_r(1)^{a_r} u$ where $u \in \mathbb{C}\{\{x_1(1), \ldots, x_r(1)\}\}$ is a unit, obtaining the conclusions of the lemma when $t = r$.

Now suppose that $l + r \ge t > r$, $g \in \mathbb{C}\{\{x_1, \ldots, x_t\}\}$ and the lemma is true in $\mathbb{C}\{\{x_1, \ldots, x_{t-1}\}\}$. We may then assume that $g \in \mathbb{C}\{\{x_1, \ldots, x_t\}\} \setminus \mathbb{C}\{\{x_1, \ldots, x_{t-1}\}\}$. Expand

$$g = \sum_{i=0}^{\infty} \sigma_i x_t^i \text{ with } \sigma_i \in \mathbb{C}\{\{x_1, \ldots, x_{t-1}\}\}.$$

Suppose that $\sigma_0, \ldots, \sigma_k$ generate the ideal $I = (\sigma_i \mid i \in \mathbb{N})$. By induction on t, there exists a sequence of transformations of types 2) - 4) $(x, y) \to (x(1), y(1))$ (with $\overline{m} \le t - r - 1$ in transformations of types 3) and 4)) such that for $0 \le i \le k$, either $\sigma_i = 0$ or

$$\sigma_i = x_1(1)^{a_1^i} \cdots x_r(1)^{a_r^i} \overline{u}_i$$

for some $a_j^i \in \mathbb{N}$ and unit $\overline{u}_i \in \mathbb{C}\{\{x_1(1), \ldots, x_{t-1}(1)\}\}$. Then after a transformation of type 2) (which we incorporate into $(x, y) \to (x(1), y(1))$), we obtain (by Lemma 5.5) that

$I\mathcal{O}^{\rm an}_{X(1),e_{X(1)}}$ is principal and generated by $x_1(1)^{a_1^i}\cdots x_r(1)^{a_r^i}$ for some i. Then we have an expression

$$g=x_1(1)^{a_1}\cdots x_r(1)^{a_r}F$$

where $F\in\mathbb{C}\{\{x_1(1),\ldots,x_t(1)\}\}$ and $h:=\mathrm{ord}\,F(0,\ldots,0,x_t(1))<\infty$. If $h=0$ we have the conclusions of the lemma, so suppose that $h>0$. By Lemma 5.8, there exists a change of variables in $x_t(1)$ (inducing a transformation of type 3) with $\overline{m}=t-r$) such that F has an expression

$$F=\tau_0 x_t(1)^h+\tau_2 x_t(1)^{h-2}+\cdots+\tau_h \tag{32}$$

with $\tau_0\in\mathbb{C}\{\{x_1(1),\ldots,x_t(1)\}\}$ a unit and $\tau_i\in\mathbb{C}\{\{x_1(1),\ldots,x_{t-1}(1)\}\}$ for $2\le i\le h$. By induction on t, we can perform a sequence of transformations of types 2) - 4) $(x(1),y(1))\rightarrow(x(2),y(2))$ (with $\overline{m}\le t-r-1$ in transformations of types 3) and 4)) such that for $2\le i\le h$,

$$\tau_i=x_1(2)^{a_1^i}\cdots x_r(2)^{a_r^i}\overline{\tau}_i$$

where $\overline{\tau}_i\in\mathbb{C}\{\{x_1(2),\ldots,x_{t-1}(2)\}\}$ is either zero or a unit series. We can assume by Lemma 8.1 that $x_1(2),\ldots,x_r(2),x_t(1)$ are dependent. Now perform by Lemma 6.3 a transformation of type 4) $(x(2),y(2))\rightarrow(x(3),y(3))$ with $\overline{m}=t-r$ and substitute into (32) to get an expression

$$F=\tau_0x_1(3)^{b_1^0}\cdots x_r(3)^{b_r^0}(x_t(3)+\alpha)^h+\overline{\tau}_2x_1(3)^{b_1^2}\cdots x_r(3)^{b_r^2}(x_t(3)+\alpha)^{h-2}+\cdots+\overline{\tau}_hx_1(3)^{b_1^h}\cdots x_r(3)^{b_r^h}$$

with $0\ne\alpha\in\mathbb{C}$. Now perform a transformation of type 2) (which we incorporate into $(x(2),y(2))\rightarrow(x(3),y(3))$) to principalize the ideal

$$I=(x_1(3)^{b_1^i}\cdots x_r(3)^{b_r^i}\mid i=0\text{ or }\overline{\tau}_i\ne 0).$$

We then have an expression

$$g=x_1(3)^{\overline{a}_1}\cdots \overline{x}_r(3)^{\overline{a}_r}\overline{F}$$

where $\mathrm{ord}\,\overline{F}(0,\ldots,0,x_t(3))<h$. By induction on h, we eventually reach the conclusions of the lemma for $g\in\mathbb{C}\{\{x_1,\ldots,x_t\}\}$. The lemma now follows from induction on t. $\square$

Lemma 8.3. *Suppose that (x,y) are prepared of type (s,r,l) and $g\in\mathbb{C}\{\{y_1,\ldots,y_{s+l}\}\}$. Then either there exists a sequence of transformations $(x,y)\rightarrow(x(1),y(1))$ such that $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1)>(s,r,l)$ or there exists a sequence of transformations of the types 1) - 4) $(x,y)\rightarrow(x(1),y(1))$ such that $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1)=(s,r,l)$ and we have an expression*

$$g=y_1(1)^{d_1}\cdots y_s(1)^{d_s}u$$

with $u\in\mathbb{C}\{\{y_1(1),\ldots,y_{s+l}(1)\}\}$ a unit.

Proof. We will perform a sequence of transformations which we may assume do not lead to an increase in (s,r,l).

Let g have the expression (25). Let J be the ideal in $\mathcal{O}^{\rm an}_{\tilde{Y},e_{\tilde{Y}}}$ defined by

$$J=(h_{[\Lambda]}\mid[\Lambda]\in\mathbb{Z}^s/(\mathbb{Q}^rC)\cap\mathbb{Z}^s).$$

J is generated by $h_{[\Lambda_1]},\ldots,h_{[\Lambda_t]}$ for some $[\Lambda_1],\ldots,[\Lambda_t]$. After performing a transformation of type 2) $(x,y)\rightarrow(x(1),y(1))$ we obtain expressions

$$\delta_{[\Lambda_i]}:=\frac{h_{[\Lambda_i]}}{y_1^{\lambda_1^i}\cdots y_s^{\lambda_s^i}}x_1^{w_1^i}\cdots x_r^{w_r^i}\in\mathbb{C}\{\{x_1(1)^{\frac{1}{d}},\ldots,x_r(1)^{\frac{1}{d}},x_{r+1}(1),\ldots,x_{r+l}(1)\}\}.$$

for $1 \le i \le t$ of the form of (27) by Proposition 7.3. We may choose the $w^i_1, \ldots, w^i_r \in \mathbb{N}$ so that

$$\frac{x_1^{w_1^i} \cdots x_r^{w_r^i}}{y_1^{\lambda_1^i} \cdots y_s^{\lambda_s^i}} \in \mathbb{C}\{\{y_1, \ldots, y_s\}\}.$$

Let ω be a complex primitive d-th root of unity, and for $1 \le j \le t$, let

$$\varepsilon_{[\Lambda_j]} = \prod_{i_1,\ldots,i_r=1}^{d} \delta_{[\Lambda_j]}(\omega^{i_1} x_1(1)^{\frac{1}{d}}, \ldots, \omega^{i_r} x_r(1)^{\frac{1}{d}}, x_{r+1}(1), \ldots, x_{r+l}(1)) \in \mathbb{C}\{\{x_1(1), \ldots, x_{r+l}(1)\}\}.$$

Let

$$f = \prod_{i=1}^{t} \varepsilon_{[\Lambda_i]}.$$

By Lemma 8.2, there exists a sequence of transformations of types 2) - 4) $(x(1), y(1)) \to (x(2), y(2))$ such that $f = x_1(2)^{m_1} \cdots x_r(2)^{m_r} u$ where $u \in \mathbb{C}\{\{x_1(2), \ldots, x_{r+l}(2)\}\}$ is a unit series. Thus each $\varepsilon_{[\Lambda_i]}$ has such a form, so

$$\varepsilon_{[\Lambda_i]} = x_1(2)^{m_1^i} \cdots x_r(2)^{m_r^i} u_i$$

for $1 \le i \le t$ where $u_i \in \mathbb{C}\{\{x_1(2), \ldots, x_{r+l}(2)\}\}$ is a unit.

Let K be the quotient field of $R = \mathbb{C}\{\{y_1(2), \ldots, y_{s+l}(2)\}\}$. We have

$$\chi_{[\Lambda_i]} := \frac{\varepsilon_{[\Lambda_i]}}{\delta_{[\Lambda_i]}} \in K$$

for $1 \le i \le t$. We also have

$$\chi_{[\Lambda_i]} \in \mathbb{C}\{\{x_1(2)^{\frac{1}{d}}, \ldots, x_r(2)^{\frac{1}{d}}, x_{r+1}(2), \ldots, x_{r+l}(2)\}\},$$

as we have only performed transformations of types 2) - 4). So $\chi_{[\Lambda_i]}$ is integral over $\mathbb{C}\{\{x_1(2), \ldots, x_{r+l}(2)\}\}$ and thus $\chi_{[\Lambda_i]}$ is integral over R. Since R is a regular local ring it is normal so $\chi_{[\Lambda_i]} \in R$. Thus $\delta_{[\Lambda_i]}$ divides $\varepsilon_{[\Lambda_i]}$ in R and so there are expressions

$$\delta_{[\Lambda_i]} = y_1(2)^{e_1^i} \cdots y_s(2)^{e_s^i} v_i$$

for $1 \le i \le t$ where $v_i \in \mathbb{C}\{\{y_1(2), \ldots, y_{s+l}(2)\}\}$ are unit series and thus

$$h_{[\Lambda_i]} = y_1(2)^{m_1^i} \cdots y_s(2)^{m_s^i} \overline{u}_i$$

for $1 \le i \le t$ where $\overline{u}_i \in \mathbb{C}\{\{y_1(2), \ldots, y_{s+l}(2)\}\}$ are unit series. Now perform a transformation of type 1) to principalize the ideal $J\mathcal{O}^{\mathrm{an}}_{Y(2), e_{Y(2)}} = (y_1(2)^{m_1^i} \cdots y_s(2)^{m_s^i} \mid 1 \le i \le t)$. Then we have the desired conclusion for g by (29) in Lemma 7.5. $\square$

Lemma 8.4. *Suppose that (x, y) are prepared of type (s, r, l) and $g \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$. Then either there exists a sequence of transformations $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) > (s, r, l)$ or there exists a sequence of transformations of the types 1) - 4) and 8) $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) = (s, r, l)$ and either g is algebraic over $x_1(1), \ldots, x_{r+l}(1)$ or*

$$g = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s}$$

with $P \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ algebraic over $x_1(1), \ldots, x_{r+l}(1)$ and $y_1(1)^{d_1} \cdots y_s(1)^{d_s}$ not algebraic over $x_1(1), \ldots, x_r(1)$.

Proof. We will perform a sequence of transformations which we may assume do not lead to an increase in (s,r,l). Let g have the expression (25) and let $g' = g - h_{[0]} \in \mathbb{C}\{\{y_1,\ldots,y_{s+l}\}\}$. By Lemma 8.3, there exists a sequence of transformations of type 1) - 4) $(x,y) \to (x(1),y(1))$ so that $g' = y_1(1)^{d_1}\cdots y_s(1)^{d_s}u$ with $u \in \mathbb{C}\{\{y_1(1),\ldots,y_{s+l}(1)\}\}$ a unit. By Proposition 7.3, after possibly performing another transformation of type 2), we also obtain that $h_{[0]} \in \mathbb{C}\{\{x_1(1)^{\frac{1}{d}},\ldots,x_r(1)^{\frac{1}{d}},x_{r+1}(1),\ldots,x_{r+l}(1)\}\}$. Since $h_{[0]} \in \mathbb{C}\{\{y_1(1),\ldots,y_{s+l}(1)\}\}$, by Lemma 7.5, we have that $h_{[0]}$ is algebraic over $x_1(1),\ldots,x_{r+l}(1)$. We also have by Lemma 7.5 that

$$\operatorname{rank}\begin{pmatrix} c_{11}(1) & \cdots & c_{1s}(1) \\ \vdots & & \vdots \\ c_{r1}(1) & \cdots & c_{rs}(1) \\ d_1 & \cdots & d_s \end{pmatrix} = r+1.$$

Thus there exist $e_1,\ldots,e_s \in \mathbb{Q}$ such that

$$e_1c_{i1}(1)+\cdots+e_sc_{is}(1) = 0 \text{ for } 1 \le i \le r$$

and

$$e_1d_1+\cdots+e_sd_s = -1,$$

and so, making a change of variables, replacing $y_i(1)$ with $y_i(1)u^{e_i}$ for $1 \le i \le s$, we have a transformation of type 8) which gives the conclusions of the lemma.

□

Proposition 8.5. *Suppose that (x,y) are prepared of type (s,r,l). Then either there exists a sequence of transformations $(x,y) \to (x(1),y(1))$ such that $(x(1),y(1))$ are prepared of type (s_1,r_1,l_1) with $(s_1,r_1,l_1) > (s,r,l)$ or the induced homomorphism*

$$\alpha : \mathbb{C}[[x_1,\ldots,x_{r+l},x_{r+l+1}]] \to \mathbb{C}[[y_1,\ldots,y_n]]$$

is an injection.

Proof. Set $z = \alpha(x_{r+l+1})$ and suppose that there exists a nonzero series $G \in \mathbb{C}[[x_1,\ldots,x_{r+l+1}]]$ such that $\alpha(G) = 0$. Expand G as

$$G = \sum_{i=0}^{\infty} a_i(x_1,\ldots,x_{r+l})x_{r+l+1}^i$$

with $a_i(x_1,\ldots,x_{r+l}) \in \mathbb{C}[[x_1,\ldots,x_{r+l}]]$ for all i. We have $\alpha(a_i) \in \mathbb{C}[[y_1,\ldots,y_{s+l}]]$ for all i and

$$0 = \alpha(G) = \sum_i \alpha(a_i)z^i = 0 \tag{33}$$

in $\mathbb{C}[[y_1,\ldots,y_n]]$.

Let $A = \mathbb{C}[[y_1,\ldots,y_{s+l}]]$ and $A[[t]]$ be a power series ring in one variable. Let

$$f(t) = \sum \alpha(a_i)t^i \in A[[t]].$$

$f(t)$ is nonzero since $\alpha(a_i)$ is nonzero whenever a_i is nonzero.

Suppose that $z \notin \mathbb{C}[[y_1,\ldots,y_{s+l}]]$. We will derive a contradiction. Expand

$$z = \sum b_{i_{s+l+1},\ldots,i_n} y_{s+l+1}^{i_{s+l+1}}\cdots y_n^{i_n}$$

in $\mathbb{C}[[y_1,\ldots,y_n]]$ with $b_{i_{s+l+1},\ldots,i_n} \in \mathbb{C}[[y_1,\ldots,y_{s+l}]]$. Since z is in the maximal ideal of $\mathbb{C}[[y_1,\ldots,y_n]]$, we have that $b_{0,\ldots,0}$ is in the maximal ideal of $\mathbb{C}[[y_1,\ldots,y_{s+l}]]$. Thus the

map $g(t) \mapsto g(t + b_{0,\ldots,0})$ is an isomorphism of $A[[t]]$. Let $\overline{f}(t) = f(t + b_{0,\ldots,0})$. We have that $\overline{f}(t) \neq 0$. Let $\overline{z} = z - b_{0,\ldots,0}$. We have that $\overline{f}(\overline{z}) = 0$. Let $(j_{s+l+1}, \ldots, j_n)$ be the minimum in the lex order of

$$\{(i_{s+l+1}, \ldots, i_n) \mid b_{i_{s+l+1},\ldots,i_n} \neq 0 \text{ and } (i_{s+l+1}, \ldots, i_n) \neq (0, \ldots, 0)\}.$$

Then $\overline{f}(\overline{z})$ has a nonzero $\lambda(j_{s+l+1}, \ldots, j_n)$ term, where λ is the smallest positive exponent of t such that $\overline{f}(t)$ has a nonzero t^{λ} term. This contradiction shows that $z \in \mathbb{C}[[y_1, \ldots, y_{s+l}]]$. Thus

$$z \in \mathbb{C}[[y_1, \ldots, y_{s+l}]] \cap \mathbb{C}\{\{y_1, \ldots, y_n\}\} = \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}.$$

Suppose that z is not algebraic over $x_1, \ldots, x_{r+l}$ (Definition 7.1). Let

$$z = \sum_{[\Lambda]\in(\mathbb{Z}^s/(\mathbb{Q}^r C)\cap\mathbb{Z}^s)} h_{[\Lambda]}$$

be the decomposition of (25). Then $z \neq h_{[0]}$ since we are assuming that z is not algebraic over $x_1, \ldots, x_{r+l}$. Since z is in the maximal ideal of $\mathbb{C}[[y_1, \ldots, y_n]]$ we have that $h_{[0]}$ is in the maximal ideal of $\mathbb{C}[[y_1, \ldots, y_{s+l}]]$. Thus the map $g(t) \mapsto g(t + h_{[0]})$ is an isomorphism of $A[[t]]$. Let $\tilde{f}(t) = f(t + h_{[0]})$. We have that $\tilde{f}(t) \neq 0$. Further, all coefficients of $\tilde{f}(t)$ are algebraic over $x_1, \ldots, x_{r+l}$.

Let $\tilde{z} = z - h_{[0]} \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$. We have that $\tilde{f}(\tilde{z}) = 0$. By Lemma 8.3, there either exists a sequence of transformations $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) > (s, r, l)$ or there exists a sequence of transformations $(x, y) \to (x(1), y(1))$ of types 1) - 4) such that we have an expression

$$\tilde{z} = y_1(1)^{d_1} \cdots y_s(1)^{d_s} u$$

where $u \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ is a unit. We have that

$$\operatorname{rank} \begin{pmatrix} c_{11}(1) & \cdots & c_{1s}(1) \\ \vdots & & \vdots \\ c_{r1}(1) & \cdots & c_{rs}(1) \\ d_1 & \cdots & d_s \end{pmatrix} = r + 1$$

by Lemma 7.5. We now perform a transformation of type 8), replacing $y_i(1)$ with $y_i(1)u^{\lambda_i}$ for some $\lambda_i \in \mathbb{Q}$ for $1 \leq i \leq s$ to obtain that $\tilde{z} = y_1(1)^{d_1} \cdots y_s(1)^{d_s}$. We have that $\tilde{f}(t) \in A_1[[t]]$ where $A_1 = \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ and all coefficients e_i of

$$\tilde{f}(t) = \sum e_i t^i$$

are algebraic over $x_1(1), \ldots, x_{r+l}(1)$ by Lemma 5.2. From the expansion

$$0 = \tilde{f}(\tilde{z}) = \sum_{i=0}^{\infty} e_i (y_1(1)^{d_1} \cdots y_s(1)^{d_s})^i$$

we see that this is the expansion of type (25) of $\tilde{f}(\tilde{z}) = 0$, so that $e_i(y_1(1)^{d_1} \cdots y_s(1)^{d_s})^i = 0$ for all i, which implies that $e_i = 0$ for all i so that $\tilde{f}(t) = 0$, giving a contradiction, so z is algebraic over $x_1, \ldots, x_{r+l}$. By Lemma 7.4, identifying z with x_{r+l+1} by the inclusion $\mathbb{C}\{\{x_1, \ldots, x_n\}\} \subset \mathbb{C}\{\{y_1, \ldots, y_n\}\}$, we have that $x_{r+l+1} \in \mathbb{C}\{\{x_1, \ldots, x_{r+l}\}\}$, a contradiction. Thus α is injective. □

Lemma 8.6. *Suppose that (x, y) are prepared of type (s, r, l) and $g \in \mathbb{C}\{\{y_1, \ldots, y_t\}\}$ with $s + l \leq t \leq n$. Then either there exists a sequence of transformations $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) > (s, r, l)$ or there exists a sequence of transformations of the types 1) - 6) $(x, y) \to (x(1), y(1))$ (with $l < \overline{m} \leq t - s$ in transformations of type 5) - 6)) such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) = (s, r, l)$ and*

$$g = y_1(1)^{d_1} \cdots y_s(1)^{d_s} u$$

with $u \in \mathbb{C}\{\{y_1(1), \ldots, y_t(1)\}\}$ a unit.

Proof. We will perform a sequence of transformations which we may assume do not lead to an increase in (s, r, l). The proof is by induction on t with $s + l \leq t \leq n$, with $g \in \mathbb{C}\{\{x_1, \ldots, x_t\}\}$. The case $t = s + l$ is proven in Lemma 8.3. Thus we may assume that $t > s + l$. Write

$$g = \sum \sigma_i y_t^i$$

where $\sigma_i \in \mathbb{C}\{\{y_1, \ldots, y_{t-1}\}\}$. Let $I = (\sigma_i \mid i \geq 0)$. There exist $\sigma_0, \ldots, \sigma_k$ which generate I. by induction, there exist a sequence of transformations of the types 1) - 6) $(x, y) \to (x(1), y(1))$ (with $l < \overline{m} \leq t-1-s$ whenever a transformation of type 5) or 6) is performed) such that

$$\sigma_j = y_1(1)^{i_1(j)} \cdots y_s(1)^{i_s(j)} \overline{u}_j$$

for $0 \leq j \leq k$ where $\overline{u}_j \in \mathbb{C}\{\{y_1(1), \ldots, y_{t-1}(1)\}\}$ is a unit or zero. Now perform a transformation of type 1) (which we incorporate into $(x, y) \to (x(1), y(1))$) to make I principal. Then we have an expression $g = y_1(1)^{m_1} \cdots y_s(1)^{m_s} \overline{g}$ where $h = \text{ord}(\overline{g}(0, \ldots, 0, y_t(1)) < \infty$. If $h = 0$ we are done. We will now proceed by induction on h. By Lemma 5.8, we can perform a transformation of type 5), replacing $y_t(1)$ with $y_t(1) - \Phi$ for an appropriate $\Phi \in \mathbb{C}\{\{y_1(1), \ldots, y_{t-1}(1)\}\}$, to obtain an expression

$$\overline{g} = \tau_0 y_t(1)^h + \tau_1 y_t(1)^{h-2} + \cdots + \tau_h \tag{34}$$

with $\tau_0 \in \mathbb{C}\{\{y_1(1), \ldots, y_t(1)\}\}$ a unit series and $\tau_i \in \mathbb{C}\{\{y_1(1), \ldots, y_{t-1}(1)\}\}$ for $1 \leq i \leq h$. By induction on t, we may construct a sequence of transformations of type 1) - 6) $(x(1), y(1)) \to (x(2), y(2))$ (with $\overline{m} \leq t - 1 - s$ whenever a transformation of type 5) or 6) is performed) such that for $2 \leq i \leq h$, whenever τ_i is nonzero, it has an expression

$$\tau_i = y_1(2)^{j_1^i} \cdots y_s(2)^{j_s^i} \overline{u}_i$$

where $\overline{u}_i \in \mathbb{C}\{\{y_1(2), \ldots, y_{t-1}(2)\}\}$ is a unit series. Since $y_t(2)$ is dependent on $y_1(2), \ldots, y_s(2)$, there exists a transformation of type 6) $(x(2), y(2)) \to (x(3), y(3))$ with $\overline{m} = t$, which we perform. Substituting into (34), we obtain

$$\overline{g} = \tau_0 y_1(3)^{b_1^0} \cdots y_s(3)^{b_s^0}(y_t(3)+\alpha)^h + y_1(3)^{b_1^2} \cdots y_s(3)^{b_s^2} \overline{u}_2 (y_t(3)+\alpha)^{h-2} + \cdots + y_1(3)^{b_1^h} \cdots y_s(3)^{b_s^h} \overline{u}_h$$

(with $0 \neq \alpha \in \mathbb{C}$). Now perform a transformation of type 1) (which we incorporate into $(x(2), y(2)) \to (x(3), y(3))$) to principalize the ideal

$$J = (y_1(3)^{b_1^0} \cdots y_s(3)^{b_s^0}, y_1(3)^{b_1^2} \cdots y_s(3)^{b_s^2} \overline{u}_2, \ldots, y_1(3)^{b_1^h} \cdots y_s(3)^{b_s^h} \overline{u}_h),$$

giving us that $\overline{g} = y_1(3)^{d_1} \cdots y_s(3)^{d_s} \tilde{g}$ with $\text{ord}(\tilde{g}(0, \ldots, y_t(3))) < h$. By induction on h, we obtain the conclusions of the lemma.

$\square$

Lemma 8.7. *Suppose that (x, y) are prepared of type (s, r, l) and*

$$g \in \mathbb{C}\{\{y_1, \ldots, y_t\}\} \setminus \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$$

with $s + l < t \leq n$. Then either there exists a sequence of transformations $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) > (s, r, l)$ or there exists a sequence of transformations of the types 1) - 7) $(x, y) \to (x(1), y(1))$ (with $\overline{m} \leq t - s$ in transformations of types 5) - 7)) such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) = (s, r, l)$ and

$$g = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s} y_t$$

with $P \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$.

Proof. We will perform a sequence of transformations which we may assume do not lead to an increase in (s, r, l). Write $g = \sum_{i \geq 0} \sigma_i y_t^i$ with $\sigma_i \in \mathbb{C}\{\{y_1, \ldots, y_t\}\}$. Let I be the ideal $I = (\sigma_i \mid i > 0)$. Suppose that I is generated by $\sigma_1, \ldots, \sigma_k$. By Lemma 8.6, there exist a sequence of transformations of types 1) - 6) $(x, y) \to (x(1), y(1))$ (with $\overline{m} \leq t - s - 1$ if a tranformation of type 5) or 6) is performed) such that for $1 \leq j \leq k$,

$$\sigma_j = y_1(1)^{i_1^j} \cdots y_s(1)^{i_s^j} u_j$$

with $u_j \in \mathbb{C}\{\{y_1(1), \ldots, y_{t-1}(1)\}\}$ a unit (or zero). By induction on t in Lemma 8.7, there exists a sequence of transformations of types 1) - 7) $(x(1), y(1)) \to (x(2), y(2))$ (with $\overline{m} \leq t - s - 1$ if a transformation of type 5), 6) or 7) is performed) such that

$$\sigma_0 = P_0 + y_1(2)^{a_1} \cdots y_s(2)^{a_s} y_{t-1}(2) \tag{35}$$

or

$$\sigma_0 = P_0 \tag{36}$$

with $P_0 \in \mathbb{C}\{\{y_1(2), \ldots, y_{s+l}(2)\}\}$. Case (35) can only occur if $t > s + l + 1$.

Let J be the ideal $I\mathcal{O}^{\rm an}_{Y(2), e_{Y(2)}} + (y_1(2)^{a_1} \cdots y_s(2)^{a_s})$ if (35) holds and $J = I\mathcal{O}^{\rm an}_{Y(2), e_{Y(2)}}$ if (36) holds. J is generated by monomials in $y_1(2), \ldots, y_s(2)$. There exists a transformation of type 1) $(x(2), y(2)) \to (x(3), y(3))$ such that $J\mathcal{O}^{\rm an}_{X(3), e_{X(3)}}$ is principal by Lemma 5.5, so

$$g = P_0 + \sum_{i > 0} \sigma_i y_t(2)^i + y_1(2)^{a_1} \cdots y_s(2)^{a_s} y_{t-1}(2)\overline{u} = P_0 + y_1(3)^{d_1} \cdots y_s(3)^{d_s} \overline{g}$$

where $\overline{u}$ is zero or 1, $P_0 \in \mathbb{C}\{\{y_1(3), \ldots, y_{s+l}(3)\}\}$ and $\overline{g} \in \mathbb{C}\{\{y_1(3), \ldots, y_t(3)\}\}$ is not divisible by $y_1(3), \ldots, y_s(3)$. If $\operatorname{ord} \overline{g}(0, \ldots, 0, y_{t-1}(3), 0) = 1$ we set $y_t(3) = \overline{g}$ and $y_{t-1}(3) = y_t(3)$ (a composition of transformations of type 7) and 5)) to get the conclusions of Lemma 8.7. Otherwise, we have

$$0 < \operatorname{ord} \overline{g}(0, \ldots, 0, y_t(3)) < \infty.$$

Now suppose that

$$g = P + y_1(3)^{d_1} \cdots y_s(3)^{d_s} F \tag{37}$$

where $P \in \mathbb{C}\{\{y_1(3), \ldots, y_{s+l}(3)\}\}$, $F \in \mathbb{C}\{\{y_1(3), \ldots, y_t(3)\}\}$ is such that the power series expansion of $y_1(3)^{d_1} \cdots y_s(3)^{d_s} F$ has no monomials in $y_1(3), \cdots, y_{s+l}(3)$; that is,

$$F(y_1(3), \ldots, y_{s+l}(3), 0, \ldots, 0) = 0,$$

$y_i(3) \not| F$ for $1 \leq i \leq s$ and

$$0 < h := \operatorname{ord} F(0, \ldots, 0, y_t(3)) < \infty.$$

If $h = 1$, we can set $y_t(3) = F$ (a transformation of type 5)) to get the conclusions of Lemma 8.7 for g.

Suppose that $h > 1$. By Lemma 5.8, we can make a change of variables, replacing $y_t(3)$ with $y_t(3) - \Phi$ for an appropriate $\Phi \in \mathbb{C}\{\{y_1(3), \ldots, y_{t-1}(3)\}\}$ (a transformation of type 5)) to get an expression

$$F = \tau_0 y_t(3)^h + \tau_2 y_t(3)^{h-2} + \cdots + \tau_h \tag{38}$$

where $\tau_0 \in \mathbb{C}\{\{y_1(3), \ldots, y_t(3)\}\}$ is a unit and $\tau_i \in \mathbb{C}\{\{y_1(3), \ldots, y_{t-1}(3)\}\}$ for $2 \leq i \leq h$. By Lemma 8.6, there exists a sequence of transformations of types 1) - 6) $(x(3), y(3)) \to (x(4), y(4))$ (with $\overline{m} < t-s$ for transformations of types 5) - 6)) such that for $2 \leq i \leq h-1$,

$$\tau_i = y_1(4)^{j_1^i} \cdots y_s(4)^{j_s^i} \overline{u}_i$$

with $\overline{u}_i \in \mathbb{C}\{\{y_1(4), \ldots, y_{t-1}(4)\}\}$ either a unit or zero. By induction on t in Lemma 8.7, there exists a sequence of transformations of types 1) - 7) $(x(4), y(4)) \to (x(5), y(5))$ (with $\overline{m} < t - s$ for transformations of types 5) - 7)) such that we further have that

$$\tau_h = P_0 + y_1(5)^{c_1} \cdots y_s(5)^{c_s} y_{t-1} \overline{u} \tag{39}$$

where $\overline{u}$ is zero or 1 and $P_0 \in \mathbb{C}\{\{y_1(5), \ldots, y_{s+l}(5)\}\}$. Since $y_t(5)$ is dependent on $y_1(5), \ldots, y_s(5)$, there exists a transformation of type 6) $(x(5), y(5)) \to (x(6), y(6))$ with $\overline{m} = t - s$. Perform it and substitute into (38) to get

$$\begin{aligned} F &= \tau_0 y_1(6)^{b_1^0} \cdots y_s(6)^{b_s^0} (y_t(6) + \alpha)^h + y_1(6)^{b_1^2} \cdots y_s(6)^{b_s^2} \overline{u}_2 (y_t(6) + \alpha)^{h-2} + \cdots \\ &\quad + y_1(6)^{d_1} \cdots y_s(6)^{d_s} y_{t-1}(6) \overline{u} + P_0 \end{aligned}$$

Now perform a transformation of type 1) $(x(6), y(6)) \to (x(7), y(7))$ to principalize the ideal

$$K = (y_1(6)^{b_1^0} \cdots y_s(6)^{b_s^0}) + (y_1(6)^{b_1^i} \cdots y_s(6)^{b_s^i} \mid \overline{u}_i \neq 0) + (\overline{u} y_1(1)^{d_1} \cdots y_s(1)^{d_s}).$$

We obtain an expression

$$g = P_1 + y_1(7)^{e_1} \cdots y_s(7)^{e_s} \overline{F}$$

where

$$P_1 = P + y_1(3)^{d_1} \cdots y_s(3)^{d_s} F(y_1(7), \ldots, y_{s+l}(7), 0, \ldots, 0) \in \mathbb{C}\{\{y_1(7), \ldots, y_{s+l}(7)\}\}$$

and

$$y_1(7)^{e_1} \cdots y_s(7)^{e_s} \overline{F} = y_1(3)^{d_1} \cdots y_s(3)^{d_s} (F - F(y_1(7), \ldots, y_s(7), 0, \ldots, 0))$$

is such that $y_i(7) \not| \overline{F}$ for $1 \leq i \leq s$. We either have $\mathrm{ord}\ \overline{F}(0, \ldots, 0, y_{t-1}(7), 0) = 1$ or $1 \leq \mathrm{ord}\ \overline{F}(0, \ldots, 0, y_t(7)) < h$. In the first case, set $y_t(7) = F$ and $y_{t-1}(7) = y_t(7)$ (a composition of transformations of type 7) and 5)) to get the conclusions of Lemma 8.7. Otherwise we have a reduction in h in (37). By induction in h we will eventually get the conclusions of Lemma 8.7. □

Proposition 8.8. *Suppose that (x, y) are prepared of type (s, r, l) with $r + l < m$. Then either there exists a sequence of transformations $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) > (s, r, l)$ or there exists a sequence of transformations of types 1) - 8) $(x, y) \to (x(1), y(1))$ such that $(x(1), y(1))$ are prepared of type (s_1, r_1, l_1) with $(s_1, r_1, l_1) = (s, r, l)$ and we have an expression*

$$x_{r+l+1}(1) = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s} \tag{40}$$

with $P \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ *algebraic over* $x_1(1), \ldots, x_{r+l}(1)$ *and* $y_1(1)^{d_1} \cdots y_s(1)^{d_s}$ *not algebraic over* $x_1(1), \ldots, x_r(1)$ *or we have an expression*

$$x_{r+l+1}(1) = P + y_1(1)^{d_1} \cdots y_s(1)^{d_s} y_{s+l+1}(1) \tag{41}$$

with $P \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$ *algebraic over* $x_1(1), \ldots, x_{r+l}(1)$.

Proof. We will construct a sequence of transformations such that either we obtain an increase in (s, r, l), or we obtain the conclusions of Proposition 8.8. We may thus assume that all transformations in the course of our proof do not give an increase in (s, r, l).

We have that x_{r+l+1} is not algebraic over $x_1, \ldots, x_{r+l}$ by Lemma 7.4.

First suppose that $x_{r+l+1} \in \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$. Then there exists a sequence of transformations of types 1) - 4) and 8) such that the conclusions of Lemma 8.4 hold, giving an expression (40) of the conclusions of Proposition 8.8, since x_{r+l+1} is not algebraic over $x_1(1), \ldots, x_{r+l}(1)$ by Lemma 7.5.

Now suppose that $x_{r+l+1} \notin \mathbb{C}\{\{y_1, \ldots, y_{s+l}\}\}$. Then by Lemma 8.7, there exists a sequence of transformations of types 1) - 7) $(x, y) \to (x(1), y(1))$ such that we have an expression

$$x_{r+l+1}(1) = \tilde{P} + y_1(1)^{a_1} \cdots y_s(1)^{a_s} y_{s+l+1}(1) \tag{42}$$

with $\tilde{P} \in \mathbb{C}\{\{y_1(1), \ldots, y_{s+l}(1)\}\}$. Then by Lemma 8.4, there exists a sequence of transformations 1) - 4) and 8) $(x(1), y(1)) \to (x(2), y(2))$ such that we have an expression (42) with

$$\tilde{P} = P' + y_1(2)^{b_1} \cdots y_s(2)^{b_s} \overline{u}$$

where P' is algebraic over $x_1(2), \ldots, x_{r+l}(2)$ and $y_1(2)^{b_1} \cdots y_s(2)^{b_s}$ is not algebraic over $x_1(2), \ldots, x_r(2)$ and $\overline{u}$ is 0 or 1. If $\overline{u} = 0$ we have achieved the conclusions of (41) of Proposition 8.8, so assume that $\overline{u} = 1$. Now (by Lemma 5.5) perform a transformation of type 1) $(x(2), y(2)) \to (x(3), y(3))$ to principalize the ideal

$$L = (y_1(2)^{b_1} \cdots y_s(2)^{b_s}, y_1(1)^{a_1} \cdots y_s(1)^{a_s}).$$

If $y_1(2)^{b_1} \cdots y_s(2)^{b_s}$ divides $y_1(1)^{a_1} \cdots y_s(1)^{a_s}$ (in $\mathcal{O}^{\mathrm{an}}_{X(3), e_{X(3)}}$), since we have the condition that $y_1(2)^{b_1} \cdots y_s(2)^{b_s}$ is not algebraic over $x_1(3), \ldots, x_s(3)$ from Lemma 7.5, we can change variables, multiplying the y_i by units for $1 \leq i \leq s$ to get an expression (40) of the conclusions of Proposition 8.8 (a transformation of type 8)). If $y_1(2)^{b_1} \cdots y_s(2)^{b_s}$ does not divide $y_1(1)^{a_1} \cdots y_s(1)^{a_s}$ in $\mathcal{O}^{\mathrm{an}}_{X(3), e_{X(3)}}$ (so that $y_1(1)^{a_1} \cdots y_s(1)^{a_s}$ properly divides $y_1(2)^{b_1} \cdots y_s(2)^{b_s}$ in $\mathcal{O}^{\mathrm{an}}_{X(3), e_{X(3)}}$) we have an expression

$$x_{r+l+1}(3) = P + y_1(3)^{\overline{a}_1} \cdots y_s(3)^{\overline{a}_s} F$$

with $F \in \mathbb{C}\{\{y_1(3), \ldots, y_s(3), y_{s+l+1}(3)\}\}$ such that ord $F(0, \ldots, 0, y_{s+l+1}(3)) = 1$. Replacing $y_{s+l+1}(3)$ with F (a transformation of type 5)) we get an expression of the form (41) of the conclusions of Proposition 8.8.

$\square$

Proposition 8.9. *Suppose that* (x, y) *are prepared of type* (s, r, l) *with* $r + l < m$. *Then there exists a sequence of transformations of types 1) - 10)* $(x, y) \to (x(1), y(1))$ *such that* $(x(1), y(1))$ *are prepared of type* (s_1, r_1, l_1) *with* $(s_1, r_1, l_1) > (s, r, l)$

Proof. We may assume that all transformations of type 1) - 10) in the course of our proof do not give an increase in (s, r, l); otherwise we have obtained the conclusions of the theorem and we can terminate our algorithms. By Proposition 8.8, there exists a sequence

of transformations of types 1) - 8) $(x,y)\rightarrow(x(0),y(0))$ such that we have an expression (for $i=0$)

$$x_{r+l+1}(i)=P+y_1(i)^{d_1}\cdots y_s(i)^{d_s} \tag{43}$$

with P algebraic over $x_1(i),\ldots,x_{r+l}(i)$ and $y_1(i)^{d_1}\cdots y_s(i)^{d_s}$ not algebraic over $x_1(i),\ldots,x_r(i)$ or we have an expression

$$x_{r+l+1}(i)=P+y_1(i)^{d_1}\cdots y_s(i)^{d_s}y_{s+l+1}(i) \tag{44}$$

with P algebraic over $x_1(i),\ldots,x_{r+l}(i)$.

We will perform sequences of transformations $(x,y)\rightarrow(x(i),y(i))$ in the course of this proof which preserve the respective expressions (43) or (44).

We will now construct a function g (in equation (46)) using transformations which preserve the respective form (43) or (44). The function g (or its strict transform) will play a major role in the proof.

The decomposition (25) of P is $P=h_{[0]}$ since P is algebraic over $x_1(1),\ldots,x_{r+l}(1)$. There exists a transformation of type 2) $(x(0),y(0))\rightarrow(x(1),y(1))$ such that

$$P\in\mathbb{C}\{\{x_1(1)^{\frac{1}{d}},\ldots,x_r(1)^{\frac{1}{d}},x_{r+1}(1),\ldots,x_{r+l}(1)\}\}$$

for some d by Proposition 7.3. Let ω be a primitive d-th root of unity in $\mathbb{C}$. Let

$$S_{i_1,\ldots,i_r}=P(\omega^{i_1}x_1(1)^{\frac{1}{d}},\ldots,\omega^{i_r}x_r(1)^{\frac{1}{d}},x_{r+1}(1),\ldots,x_{r+l}(1))$$

for $1\le i_1,\ldots,i_r\le d$. We have that

$$S_{i_1,\ldots,i_r}\in\mathbb{C}\{\{y_1(1)^{\frac{1}{d}},\ldots,y_s(1)^{\frac{1}{d}},y_{s+1}(1),\ldots,y_{s+l}(1)\}\}$$

for all $i_1,\ldots,i_r$ since

$$x_i(1)^{\frac{1}{d}}=\prod_{j=1}^{s}(y_j(1)^{\frac{1}{d}})^{c_{ij}(1)}\text{ for }1\le i\le r.$$

Since $P\in\mathbb{C}\{\{y_1(1),\ldots,y_{s+l}(1)\}\}$, we have that $S_{i_1,\ldots,i_r}\in\mathbb{C}\{\{y_1(1),\ldots,y_{s+l}(1)\}$ for all $i_1,\ldots,i_r$. Further, $S_{i_1,\ldots,i_r}$ is algebraic over $x_1(1),\ldots,x_{r+l}(1)$ for all $i_1,\ldots,i_r$ since P is. Let

$$R=\prod_{i_1,\ldots,i_r=1}^{d}S_{i_1,\ldots,i_r}\in\mathbb{C}\{\{x_1(1),\ldots,x_{r+l}(1)\}\}.$$

By Lemma 8.2, there exists a sequence of transformations of types 2) - 4) $(x(1),y(1))\rightarrow(x(2),y(2))$ such that

$$R=x_1(2)^{m_1}\cdots x_r(2)^{m_r}u$$

where $u\in\mathbb{C}\{\{x_1(2),\ldots,x_{r+l}(2)\}\}$ is a unit. Now P divides R in $\mathbb{C}\{\{y_1(2),\ldots,y_{r+l}(2)\}\}$, so we have that

$$P=y_1(2)^{m_1}\cdots y_s(2)^{m_s}\tilde{u} \tag{45}$$

where $\tilde{u}\in\mathbb{C}\{\{y_1(2),\ldots,y_{s+l}(2)\}\}$ is a unit and by Lemma 7.5 and since P is algebraic over $x_1(2),\ldots,x_{r+l}(2)$, we have that $y_1(2)^{m_1}\cdots y_s(2)^{m_s}$ is algebraic over $x_1(2),\ldots,x_r(2)$. Set

$$g=\prod_{i_1,\ldots,i_r=1}^{d}(x_{r+l+1}(2)-S_{i_1,\ldots,i_r})\in\mathbb{C}\{\{x_1(2),\ldots,x_{r+l+1}(2)\}\}. \tag{46}$$

Let

$$t = \mathrm{ord}\ g(0, \ldots, 0, x_{r+l+1}(2)). \tag{47}$$

We have that $0 < t \le d^r$.

The proof now proceedes by induction on t. We will make a series of transformations which will either give an increase in (s, r, l) (establishing the proposition) or preserve the respective form (43) or (44) with a reduction in t (which will remain positive) in the strict transform of g. We first perform transformations which preserve the respective forms (43) or (44) and preserve $t = \mathrm{ord}\ g(0, \ldots, 0, x_{r+l+1}(i))$ which put g into the good form of equation (49) (in case (43)) or in the good form of equation (50) (in case (44)).

Set

$$Q_{i_1,\ldots,i_r} = P - S_{i_1,\ldots,i_r}$$

which are algebraic over $x_1(2), \ldots, x_{r+l}(2)$. By the argument leading to (45), we can construct a sequence of transformations of types 2) - 4) $(x(2), y(2)) \to (x(3), y(3))$ which preserve the expressions (45), (46), t in equation (47) and the expression (43) or (44) (in the variables $x(3)$ and $y(3)$) such that for all $I = (i_1, \ldots, i_r)$,

$$Q_I = y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I} u_I \tag{48}$$

where $u_I \in \mathbb{C}\{\{y_1(3), \ldots, y_{s+l}(3)\}\}$ are units and $y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I}$ are algebraic over $x_1(3), \ldots, x_r(3)$. After a transformation of type 1) $(x(3), y(3)) \to (x(4), y(4))$, we can principalize the ideals $(y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I}, y_1(0)^{d_1} \cdots y_s(0)^{d_s})$ for all I (by Lemma 5.5), giving us the possibilities

$$x_{r+l+1}(4) - S_{i_1,\ldots,i_r} = y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I} \overline{u}_I$$

where $\overline{u}_I \in \mathbb{C}\{\{y_1(4), \ldots, y_{s+l}(4)\}\}$ is a unit and $y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I}$ is algebraic over $x_1(4), \ldots, x_r(4)$ or

$$x_{r+l+1}(4) - S_{i_1,\ldots,i_r} = y_1(0)^{d_1} \cdots y_s(0)^{d_s} \overline{u}_I$$

where $\overline{u}_I \in \mathbb{C}\{\{y_1(4), \ldots, y_{s+l}(4)\}\}$ is a unit and $y_1(0)^{d_1} \cdots y_s(0)^{d_s}$ is not algebraic over $x_1(4), \ldots, x_r(4)$ if (43) holds and giving us the possibilities

$$x_{r+l+1}(4) - S_{i_1,\ldots,i_r} = y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I} G^I$$

where $G^I \in \mathbb{C}\{\{y_1(4), \ldots, y_{s+l+1}(4)\}\}$ is a unit and $y_1(3)^{n_1^I} \cdots y_s(3)^{n_s^I}$ is algebraic over $x_1(4), \ldots, x_r(4)$ or

$$x_{r+l+1}(4) - S_{i_1,\ldots,i_r} = y_1(4)^{m_1^I} \cdots y_s(4)^{m_s^I} G^I$$

where $G^I \in \mathbb{C}\{\{y_1(4), \ldots, y_{s+l+1}(4)\}\}$ satisfies $\mathrm{ord}\ G^I(0, \ldots, 0, y_{r+l+1}(4)) = 1$ if (44) holds. We have that

$$x_{r+l+1}(4) - P = y_1(0)^{d_1} \cdots y_s(0)^{d_s}$$

in case (43) and

$$x_{r+l+1}(4) - P = y_1(0)^{d_1} \cdots y_s(0)^{d_s} y_{s+l+1}(4)$$

in case (44). We thus have

$$g = y_1(4)^{m_1} \cdots y_s(4)^{m_s} u \tag{49}$$

where $u \in \mathbb{C}\{\{y_1(4), \ldots, y_{s+l}(4)\}\}$ is a unit and $y_1(4)^{m_1} \cdots y_s(4)^{m_s}$ is not algebraic over $x_1(4), \ldots, x_r(4)$ in case (43) and

$$g = y_1(4)^{m_1} \cdots y_s(4)^{m_s} y_{s+l+1}(4) \prod_{I \ne (d,\ldots,d)} G^I \tag{50}$$

where for all I, $G^I \in \mathbb{C}\{\{y_1(4),\ldots,y_{s+l+1}(4)\}\}$ satisfies ord $G^I(0,\ldots,0,y_{s+l+1}(4)) = 1$ or 0 in case (44).

We now consider a special case, when g has an expression of the form (51) below, and show that after a few transformations we obtain the conclusions of the proposition.

Suppose that there exists $\Phi \in \mathbb{C}\{\{x_1(4),\ldots,x_{r+l}(4)\}\}$ such that

$$g = \tilde{u}(x_{r+l+1}(4) - \Phi)^{\lambda} \tag{51}$$

where $\lambda \in \mathbb{Z}_{>0}$ and $\tilde{u} \in \mathbb{C}\{\{x_1(4),\ldots,x_{r+l+1}(4)\}\}$ is a unit series. Setting $P' = P - \Phi$, we have an expression

$$x_{r+l+1}(4) - \Phi = P' + Q$$

where $Q := x_{r+l+1}(4) - P$ has the expression

$$Q = \begin{cases} y_1(0)^{d_1}\cdots y_s(0)^{d_s} \text{ of (43) or} \\ y_1(0)^{d_1}\cdots y_s(0)^{d_s} y_{s+l+1}(4) \text{ of (44)} \end{cases}$$

and P' is algebraic over $x_1(4),\ldots,x_{r+l}(4)$. By Lemma 8.3, there exists a sequence of transformations of types 1) - 4) $(x(4),y(4)) \to (x(5),y(5))$ such that

$$P' = y_1(5)^{a_1}\cdots y_s(5)^{a_s} u'$$

where $u' \in \mathbb{C}\{\{y_1(5),\ldots,y_{s+l}(5)\}\}$ is a unit series. We have that $y_1(5)^{a_1}\cdots y_s(5)^{a_s}$ is algebraic over $x_1(5),\ldots,x_r(5)$ by Lemma 7.5. By Lemma 5.5, after a transformation of type 1), $(x(5),y(5)) \to (x(6),y(6))$, we have that in the case when (43) holds,

$$x_{r+l+1}(6) - \Phi = y_1(6)^{n_1}\cdots y_s(6)^{n_s}\hat{u} \tag{52}$$

with $\hat{u} \in \mathbb{C}\{\{y_1(6),\ldots,y_{s+l+1}(6)\}\}$ a unit and in the case when (44) holds, we have

(53)

$$x_{r+l+1}(6) - \Phi = \begin{cases} y_1(6)^{n_1}\cdots y_s(6)^{n_s}\hat{u} & \text{with } \hat{u} \in \mathbb{C}\{\{y_1(6),\ldots,y_{s+l+1}(6)\}\} \text{ a unit} \\ & \text{and } y_1(6)^{n_1}\cdots y_s(6)^{n_s} \text{ algebraic over} \\ & x_1(6),\ldots,x_{r+l}(6), \text{ or} \\ y_1(6)^{n_1}\cdots y_s(6)^{n_s}F & \text{with } F \in \mathbb{C}\{\{y_1(6),\ldots,y_{s+l+1}(6)\}\} \\ & \text{such that ord } F(0,\ldots,0,y_{s+l+1}(6)) = 1. \end{cases}$$

If Case (43) holds, we have from comparison of the equations (52), (49) and (51) that

$$y_1(4)^{m_1}\cdots y_s(4)^{m_s} = (y_1(6)^{n_1}\cdots y_s(6)^{n_s})^{\lambda}$$

where $y_1(4)^{m_1}\cdots y_s(4)^{m_s}$ is not algebraic over $x_1(6),\ldots,x_{r+l}(6)$. Thus $y_1(6)^{n_1}\cdots y_s(6)^{n_s}$ is also not algebraic over $x_1(6),\ldots,x_{r+l}(6)$. Making a change of variables replacing $x_{r+l+1}(6)$ with $x_{r+l+1}(6) - \Phi$ and $y_1(6),\ldots,y_s(6)$ with their products by appropriate units in $\mathbb{C}\{\{y_1(6),\ldots,y_{s+l}(6)\}\}$ (transformations of types 10) and 8)), we get

$$x_{r+l+1}(6) = y_1(6)^{n_1}\cdots y_s(6)^{n_s}$$

with $y_1(6)^{n_1}\cdots y_s(6)^{n_s}$ not algebraic over $x_1(6),\ldots,x_r(6)$ obtaining an increase in r (and (s,r,l)), and so we have achieved the conclusions of Proposition 8.9.

If case (44) holds, then (50), (53) and (51) hold, so we have that

$$x_{r+l+1}(6) - \Phi = y_1(6)^{n_1}\cdots y_s(6)^{n_s}F$$

where $F \in \mathbb{C}\{\{y_1(6),\ldots,y_{s+l+1}(6)\}\}$ satisfies ord $F(0,\ldots,0,y_{s+l+1}(6)) = 1$. Then making changes of variables, replacing $y_{n+l+1}(6)$ with F and $x_{r+1+1}(6)$ with $x_{r+l+1}(6) - \Phi$ (transformations of types 5) and 10)), we have

$$x_{r+l+1}(6) = y_1(6)^{n_1}\cdots y_s(6)^{n_s}y_{s+l+1}(6).$$

If $y_1(6), \ldots, y_s(6), y_{s+l+1}(6)$ are independent, we have an increase in s (and (s, r, l)). Otherwise, we perform a SGMT in $y_1(6), \ldots, y_s(6), y_{s+l+1}(6)$ giving a transformation of type 6) $(x(6), y(6)) \to (x(7), y(7))$ such that

$$x_{r+l+1}(7) = y_1(7)^{b_1} \cdots y_s(7)^{b_s}(y_{s+l+1}(7) + \alpha)$$

for some $0 \neq \alpha \in \mathbb{C}$. If $y_1(7)^{b_1} \cdots y_s(7)^{b_s}$ is not algebraic over $x_1(7), \ldots, x_{r+l}(7)$, then we can make a change of variables in $y_1(7), \ldots, y_s(7)$, (a transformation of type 8) $(x(7), y(7)) \to (x(8), y(8))$), giving an expression

$$x_{r+l+1}(8) = y_1(8)^{b_1} \cdots y_s(8)^{b_s},$$

thus giving an increase in r (and (s, r, l)). If $y_1(7)^{b_1} \cdots y_s(7)^{b_s}$ is algebraic over $x_1(7), \ldots, x_{r+l}(7)$, then $\nu_e(x_{r+l+1}(7))$ is rationally dependent on $\nu_e(x_1(7)), \ldots, \nu_e(x_{r+l}(7))$, and so

$$x_1(7), \ldots, x_{r+l}(7), x_{r+l+1}(7)$$

are dependent by Lemma 4.1. Thus by Lemma 5.7, there exists a SGMT $(x(7)) \to (x(8))$ defined by

$$x_i(7) = \prod_{j=1}^{r} x_j(8)^{a_{ij}(8)} \text{ for } 1 \leq i \leq r \text{ and}$$

$$x_{r+l+1}(7) = \left(\prod_{j=1}^{r} x_j(8)^{a_{r+1,j}(8)}\right)(x_{r+l+1}(8) + \beta)$$

with $0 \neq \beta \in \mathbb{C}$.

By Lemma 6.4, we can extend the SGMT $(x(7)) \to (x(8))$ to a transformation $(x(7), y(7)) \to (x(8), y(8))$ of type 9) (where $(y(7)) \to (y(8))$ is a SGMT in $y_1(7), \ldots, y_s(7)$). We have

$$\left(\prod_{j=1}^{r} x_j(8)^{a_{r+1,j}(8)}\right)(x_{r+l+1}(8) + \beta) = \left(\prod_{j=1}^{s} y_j(8)^{b_j(8)}\right)(y_{r+l+1}(8) + \alpha),$$

with $\alpha, \beta \neq 0$. Then

$$\sum_{j=1}^{r} a_{r+1,j}(8)\nu_e(x_j(8)) = \sum_{j=1}^{s} b_j(8)\nu_e(y_j(8)).$$

The values $\nu_e(y_1(8)), \ldots, \nu_e(y_s(8))$ are rationally independent by Lemma 4.1, so

$$(a_{r+1,1}, \ldots, a_{r+1,r})\begin{pmatrix} c_{11}(8) & \cdots & c_{1s}(8) \\ & \vdots & \\ c_{r1}(8) & \cdots & c_{rs}(8) \end{pmatrix} = (b_1(8), \ldots, b_s(8)).$$

Thus

$$\prod_{j=1}^{r} x_j(8)^{a_{r+1.j}(8)} = \prod_{j=1}^{s} y_j(8)^{b_j(8)}$$

and $\alpha = \beta$, so $x_{r+l+1}(8) = y_{r+l+1}(8)$, giving an increase in $r + l$ (and (s, r, l)).

In all cases, we have reached the conclusions of Proposition 8.9 (under the assumption that (51) holds).

Now suppose that an expression (51) does not hold. Then $t > 1$ in (47) (by the implicit function theorem). Now we will use a Tschirnhaus transformation to put g into a good

form, and perform a sequence of transformations that preserve the respective forms (43) or (44) and lead to a decrease

$$0 < \text{ord } \overline{g}(0,\ldots,0,x_{r+l+1}(i)) < t$$

where $\overline{g}$ is the strict transform of g. The conclusions of the proposition will then follow by induction on t.

By Lemma 5.8, we can make a change of variables, replacing $x_{r+l+1}(4)$ with $x_{r+l+1}(4)-\Phi$ for some $\Phi\in\mathbb{C}\{\{x_1(4),\ldots,x_{r+l}(4)\}\}$ (a transformation of type 10)) to get an expression

$$g = \tau_0 x_{r+l+1}(4)^t + \tau_2 x_{r+l+1}(4)^{t-2} + \cdots + \tau_t \tag{54}$$

where $\tau_0\in\mathbb{C}\{\{x_1(4),\ldots,x_{r+l+1}(4)\}\}$ is a unit and $\tau_i\in\mathbb{C}\{\{x_1(4),\ldots,x_{r+l}(4)\}\}$. If all $\tau_i=0$ for $i\ge 2$ then we are in case (51), so we may suppose that some $\tau_i\ne 0$ with $i\ge 2$.

By Lemma 8.2, there exists a sequence of transformations of types 1) - 4) $(x(4),y(4))\to(x(5),y(5))$ making

$$\tau_i = x_1(5)^{a_1^i}\cdots x_s(5)^{a_s^i}\overline{u}_i$$

for $2\le i$, where $\overline{u}_i\in\mathbb{C}\{\{x_1(5),\ldots,x_{s+l}(5)\}\}$ is either a unit or zero. The forms of equations (43) and (49) or of (44) and (50) (in the variables $x(5)$ and $y(5)$) are preserved by these transformations.

Now apply the argument following (51) to $x_{r+l+1}(5)$ (in the place of $x_{r+l+1}(4)-\Phi$ in (51)) to construct a sequence of transformations of types 1) - 4) $(x(5),y(5))\to(x(6),y(6))$ to get in the case when (43) holds,

$$x_{r+l+1}(6) = y_1(6)^{n_1}\cdots y_s(6)^{n_s}\hat{u} \tag{55}$$

with $\hat{u}\in\mathbb{C}\{\{y_1(6),\ldots,y_{s+l}(6)\}\}$ a unit and in the case when (44) holds, we have

(56)

$$x_{r+l+1}(6) = \begin{cases} y_1(6)^{n_1}\cdots y_s(6)^{n_s}\hat{u} & \text{with } \hat{u}\in\mathbb{C}\{\{y_1(6),\ldots,y_{s+l+1}(6)\}\} \text{ a unit} \\ & \text{and } y_1(6)^{n_1}\cdots y_s(6)^{n_s} \text{ algebraic over } x_1(6),\ldots,x_{r+l}(6), \text{ or} \\ y_1(6)^{n_1}\cdots y_s(6)^{n_s}F & \text{with } F\in\mathbb{C}\{\{y_1(6),\ldots,y_{s+l+1}(6)\}\} \\ & \text{such that ord } F(0,\ldots,0,y_{s+l+1}(6))=1 \end{cases}$$

Suppose that (43) and (55) hold and $y_1(6)^{n_1}\cdots y_s(6)^{n_s}$ is not algebraic over $x_1(6),\ldots,x_{r+l}(6)$. Then after a transformation of type 8) we have an expression

$$x_{r+l+1}(6) = y_1(6)^{n_1}\cdots y_s(6)^{n_s}$$

giving us an increase in r (and (s,r,l)) in (14), so we have obtained the conclusions of Proposition 8.9.

Suppose that (44) and (56) hold, and we have that $x_{r+l+1}(6) = y_1(6)^{n_1}\cdots y_s(6)^{n_s}F$ with

$$\text{ord } F(0,\ldots,0,y_{n+l+1}(6)) = 1.$$

Then replacing $y_{s+l+1}(6)$ with F (a transformation of type 5)), we have relations (14) with

$$x_{r+l+1}(6) = y_1(6)^{n_1}\cdots y_s(6)^{n_s}y_{s+l+1}(6).$$

If $y_1(6),\ldots,y_s(6),y_{s+l+1}(6)$ are independent, we have an increase in s (and in (s,r,l)), and we have achieved the conclusions of Proposition 8.9, so we may suppose that

$$y_1(6),\ldots,y_s(6),y_{s+l+1}(6)$$

are dependent. If $x_1(6),\ldots,x_r(6),x_{r+l+1}(6)$ are independent, then we perform a transformation of type 6) $(x(6),y(6))\to(x(7),y(7))$ (with $\overline{m}=l+1$) to get

$$x_{r+l+1}(7) = y_1(7)^{n_1}\cdots y_s(7)^{n_s}(y_{s+l+1}(7)+\alpha)$$

with $0 \neq \alpha \in \mathbb{C}$. Since $x_1(7), \ldots, x_r(7), x_{r+l+1}(7)$ are independent (and so

$$\nu_e(x_1(7)), \ldots, \nu_e(x_r(7)), \ldots, \nu_e(x_{r+l+1}(7))$$

are rationally independent), we must have that $y_1(7)^{n_1} \cdots y_s(7)^{n_s}$ is not algebraic over $x_1(7), \ldots, x_r(7)$.

Thus after a change of variables, multiplying $y_i(7)$ by an appropriate unit for $1 \leq i \leq s$ (a transformation of type 8)), we obtain an expression (14), with an increase in r (and (s, r, l)).

The remaining case in (55) and (56) is when we have an expression

$$x_{r+l+1}(6) = y_1(6)^{\overline{m}_1} \cdots y_s(6)^{\overline{m}_s} \hat{u} \tag{57}$$

where $\hat{u} \in \mathbb{C}\{\{y_1(6), \ldots, y_{s+l+1}(6)\}\}$ is a unit and $y_1(6)^{\overline{m}_1} \cdots y_s(6)^{\overline{m}_s}$ is algebraic over $x_1(6), \ldots, x_{r+l}(6)$. We will presume that this case holds.

From (57), we see that $\nu_e(x_{r+l+1}(6))$ is rationally dependent on $\nu_e(x_1(6)), \ldots, \nu_e(x_r(6))$, so by Lemma 4.1, $x_1(6), \ldots, x_r(6), x_{r+l+1}(6)$ are dependent. Thus there exists by Lemma 5.7 a SGMT

$$\begin{array}{lcl} x_1(6) & = & x_1(7)^{a_{11}(7)} \cdots x_r(7)^{a_{1r}(7)} \\ & \vdots & \\ x_r(6) & = & x_1(7)^{a_{r1}(7)} \cdots x_r(7)^{a_{rr}(7)} \\ x_{r+l+1}(6) & = & x_1(7)^{a_1(7)} \cdots x_r(7)^{a_r(7)}(x_{r+l+1}(7) + \alpha) \end{array} \tag{58}$$

with $0 \neq \alpha \in \mathbb{C}$. Substituting into (54) and performing a (monomial) SGMT in $x_1(7), \ldots, x_r(7)$ (which we incorporate into $x(6) \to x(7)$) we obtain an expression

$$g = x_1(7)^{b_1} \cdots x_s(7)^{b_s} \overline{g}$$

where

$$\operatorname{ord} \overline{g}(0, \ldots, 0, x_{r+l+1}(7)) < t. \tag{59}$$

By Lemma 6.4, we can extend the SGMT $(x(6)) \to (x(7))$ to a transformation $(x(6), y(6)) \to (x(7), y(7))$ of type 9) (where $(y(6)) \to (y(7))$ is a SGMT in $y_1(6), \ldots, y_s(6)$).

Writing $\overline{g} = x_1(7)^{-b_1} \cdots x_s(7)^{-b_s} g$, we see from (49) or (50) that $\overline{g}$ is not a unit in $\mathbb{C}\{\{y_1(7), \ldots, y_{s+l+1}(7)\}\}$. Thus

$$\operatorname{ord} \overline{g}(0, \ldots, 0, x_{r+l+1}(7)) > 0.$$

Now $x_{r+l+1}(7)$ continues to have a form (43) or (44), and $\overline{g}$ has a form (49) (if (43) holds) or a form (50) (if (44) holds), in terms of the variables $x(7)$, $y(7)$. Thus we are in the situation after (50) (replacing g with $\overline{g}$), but by (59), we have a reduction of t in (47). By induction in t, continuing to run the algorithm following (50), we must eventually obtain the conclusions of Proposition 8.9.

□

Proposition 8.10. *Suppose that $\varphi : Y \to X$ is a morphism of complex analytic manifolds, E_Y is a simple normal crossings divisor on Y and e is an étoile over Y. Suppose that φ is quasi regular with respect to e. Then φ is regular at e_Y and there exists a commutative diagram*

$$\begin{array}{ccc} Y_e & \stackrel{\varphi_e}{\to} & X_e \\ \pi_e \downarrow & & \downarrow \lambda_e \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

of regular analytic morphisms such that the vertical arrows are products of local blow ups of nonsingular analytic subvarieties, $Y_e \to Y \in e$ and φ_e is a monomial morphism for a toroidal structure O_e on Y_e at p. Further, we have that $\pi_e^*(E_Y)$ is an effective divisor supported on O_e and the restriction of π_e to $Y_e \setminus O_e$ is an open embedding.

Proof. Let $x_1,\ldots,x_m$ be regular parameters in $\mathcal{O}^{\rm an}_{X,e_X}$ and $y_1,\ldots,y_n$ be regular parameters in $\mathcal{O}^{\rm an}_{Y,e_Y}$ such that E_Y is supported on the analytic set $Z(y_1y_2\cdots y_n)$ (in a neighborhood of e_Y in Y). After reindexing the y_i we may assume that $s\ge 1$ is such that $y_1,\ldots,y_s$ are independent and $y_1,\ldots,y_s,y_i$ are dependent for all i with $s+1\le i\le n$. After performing SGMT of type 6) for $1\le \overline{m}\le n-s$, we may assume that E_Y is supported on $Z(y_1y_2\cdots y_s)$. Then (x,y) are prepared of type $(s',0,0)$ with $s'\ge s$. By successive application of Proposition 8.9, we construct a sequence of transformations $(x,y)\to(x',y')$ such that $r'+l'=m$, giving the conclusions of the theorem.

The fact that φ_e is regular at e_{Y_e} follows from the rank theorem (page 134 [47]) and the inequality (4) applied to the monomial morphism φ_e. Thus φ is regular at e_Y as π_e and λ_e are products of local blowups, so that they are open embeddings away from nowhere dense closed analytic subspaces. □

We isolate as a corollary one of the conclusions of Proposition 8.10.

Corollary 8.11. *Suppose that $\varphi: Y\to X$ is a morphism of connected complex analytic manifolds, e is an étoile on Y and φ is quasi regular with respect to e. Then φ is regular.*

Corollary 8.11 can also be deduced from the local flattening theorem of Hironaka, Lejeune and Teissier [44] and Hironaka [42], as is shown in [27].

Theorem 8.12. *Suppose that $\varphi: Y\to X$ is a morphism of reduced complex analytic spaces, A is a closed analytic subspace of Y and e is an étoile over Y. Then there exists a commutative diagram of complex analytic morphisms*

$$\begin{array}{ccc} Y_e & \overset{\varphi_e}{\to} & X_e\\ \beta\downarrow & & \downarrow\alpha\\ Y & \overset{\varphi}{\to} & X\end{array}$$

such that $\beta\in e$, the morphisms α and β are finite products of local blow ups of nonsingular analytic sub varieties, Y_e and X_e are nonsingular analytic spaces and φ_e is a monomial analytic morphism for a toroidal structure O_e on Y_e at e_{Y_e} such that the restriction $(Y_e\setminus O_e)\to Y$ is an open embedding. There exists a nowhere dense closed analytic subspace F_e of X_e such that $X_e\setminus F_e\to X$ is an open embedding and $\varphi_e^{-1}(F_e)$ is nowhere dense in Y_e. Further, either the preimage of A in Y_e is equal to Y_e, or $\mathcal{I}_A\mathcal{O}_{Y_e}=\mathcal{O}_{Y_e}(-G)$ where $\mathcal{I}_A$ is the ideal sheaf in $\mathcal{O}_Y^{\rm an}$ of the analytic subspace A of Y and G is an effective divisor which is supported on O_e.

Proof. The proof follows by first applying Proposition 3.5 above to get a morphism of smooth analytic spaces $Y_1\to X_1$, with closed analytic sub manifold Z of X_1 such that $\varphi_1(Y_1)\subset Z$ and if $\overline{\varphi}_1: Y_1\to Z$ is the induced map, then $\overline{\varphi}_1$ is quasi regular with respect to e. We may thus replace X_1 with Z in the remainder of the proof, and assume that $\varphi_1: Y_1\to X_1$ is quasi regular with respect to e in the remainder of the proof. Either $\mathcal{I}_A\mathcal{O}_{Y_1}$ is the zero ideal sheaf, which holds if and only if the preimage of A in Y_1 is Y_1, or $\mathcal{I}_A\mathcal{O}_{Y_1}$ is a nonzero ideal sheaf. Let B be a proper closed analytic subspace of Y_1 such that $Y_1\setminus B\to Y$ is an open embedding. Then applying principalization of ideals and embedded resolution of singularities by blowing up nonsingular sub varieties to $\mathcal{I}_B$ if

$\mathcal{I}_A = (0)$ and to $\mathcal{I}_A\mathcal{I}_B$ if $\mathcal{I}_A \neq (0)$, we construct $Y_2 \to X_1$ such that either $\mathcal{I}_A\mathcal{O}_{Y_2} = (0)$ and $\mathcal{I}_B\mathcal{O}_{Y_2} = \mathcal{O}_{Y_2}(-G)$ or $\mathcal{I}_A\mathcal{I}_B\mathcal{O}_{Y_2} = \mathcal{O}_Y(-G)$ where G is a simple normal crossings divisor on Y_2 which satisfies the assumptions of Proposition 8.10, and $Y_2 \setminus G \to Y$ is an open embedding. We then apply Proposition 8.10 to $Y_2 \to X_1$ to obtain a monomial morphism at the center of e, satisfying the conclusions of the theorem. $\square$

We obtain Theorem 1.2 of the introduction as an immediate consequence of Theorem 8.12.

Theorem 8.13. *Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces, A is a closed analytic subspace of Y and $p \in Y$. Then there exists a finite number t of commutative diagrams of complex analytic morphisms*

$$\begin{array}{ccc} Y_i & \overset{\varphi_i}{\to} & X_i \\ \beta_i \downarrow & & \downarrow \alpha_i \\ Y & \overset{\varphi}{\to} & X \end{array}$$

for $1 \leq i \leq t$ such that each β_i and α_i is a finite product of local blow ups of nonsingular analytic sub varieties, Y_i and X_i are smooth analytic spaces and φ_i is a monomial analytic morphism for a toroidal structure O_i on Y_i. Either the preimage of A in Y_i is Y_i or $\mathcal{I}_A\mathcal{O}_{Y_i} = \mathcal{O}_{Y_i}(-G_i)$ where $\mathcal{I}_A$ is the ideal sheaf in $\mathcal{O}_Y^{\rm an}$ of the analytic subspace A of Y, G_i is an effective divisor which is supported on O_i, and has the further property that the restriction $(Y_i \setminus O_i) \to Y$ is an open embedding. Further, there exist compact subsets K_i of Y_i such that $\cup_{i=1}^t \beta_i(K_i)$ is a compact neighborhood of p in Y. There exist nowhere dense closed analytic subspaces F_i of X_i such that $X_i \setminus F_i \to X$ are open embeddings and $\varphi_i^{-1}(F_i)$ is nowhere dense in Y_i.

Proof. Let $\mathcal{E}_Y$ be the voûte étoilée over Y, with canonical map $P_Y : \mathcal{E}_Y \to Y$ defined by $P_Y(e) = e_Y$. We summarized in Section 3 properties of $\mathcal{E}_Y$ which we require in this proof. By Theorem 8.12, for each $e \in \mathcal{E}_Y$ we have a commutative diagram

$$\begin{array}{ccc} Y_e & \overset{\varphi_e}{\to} & X_e \\ \pi_e \downarrow & & \downarrow \\ Y & \overset{\varphi}{\to} & X \end{array}$$

such that φ_e is monomial at e_{Y_e} and satisfies the other conditions of the conclusions of Theorem 8.12. Let V_e be an open relatively compact neighborhood of e_{Y_e} in Y_e. Let $\overline{\pi}_e : V_e \to Y$ be the induced maps. Let K be a compact neighborhood of p in Y and $K' = P_Y^{-1}(K)$. The set K' is compact since P_Y is proper (Theorem 3.4 [43]). The sets $\mathcal{E}_{\overline{\pi}_e}$ (see equation (6)) give an open cover of K', so there is a finite subcover, which we reindex as $\mathcal{E}_{\overline{\pi}_{e_1}}, \ldots, \mathcal{E}_{\overline{\pi}_{e_t}}$. For $1 \leq i \leq t$, let K_i be the closure of V_{e_i} in Y_{e_i} which is compact. Since P_Y is surjective and continuous, we have inclusions of compact sets

$$p \in K \subset \cup_{i=1}^t \pi_{e_i}(K_i)$$

giving the conclusions of the theorem. $\square$

We obtain Theorem 1.3 of the introduction as an immediate consequence of Theorem 8.13.

9. Monomialization of real analytic maps

In this section we prove local monomialization theorems for real analytic morphisms. We use the method of complexifications of real analytic spaces developed in Section 1 of [42].

Remark 9.1. *Resolution of singularities of a germ of a complex analytic space (X,x), which has a natural auto conjugation, can be accomplished by blowing up smooth analytic sub varieties which are preserved by the auto conjugation. This follows by applying the basic theorem of resolution of singularities in* [40] *(or* [9]*) to the spectrum of the invariant analytic local ring Spec$((\mathcal{O}^{\rm an}_{X,x})^{\sigma})$ by the action of the auto conjugation σ of X and then extending to Spec$(\mathcal{O}^{\rm an}_{X,x})$. We also need the fact that a principalization of a sheaf of ideals which is invariant under σ can be obtained by blowing up smooth analytic sub varieties which are preserved by the auto conjugation (this also follows from* [40]*).*

Lemma 9.2. *Suppose that Y is a smooth connected real analytic variety with a complexification $\tilde Y$ which is a smooth connected complex variety. Suppose that $Z\subset Y$ is a closed real analytic subspace of Y such that its complexification $\tilde Z\subset\tilde Y$ is a nowhere dense closed complex analytic subspace of $\tilde Y$. Then Z is nowhere dense in Y (in the Euclidean topology).*

The necessity that Y be smooth in the lemma can be seen from consideration of the Whitney Umbrella $x^2-zy^2=0$.

Proof. Since $\tilde Y$ and Y are manifolds, for all $p\in Y$, the topological dimension of Y at p, T-$\dim_p Y$ (Remarks 5.16 and 5.17 [42] and Section 3), is equal to the dimension $\dim_p\tilde Y$ of $\tilde Y$ (Section 3), which is equal to $\dim\mathcal{O}^{\rm an}_{\tilde Y,p}$. Since $\tilde Z$ is a nowhere dense analytic subspace of the manifold $\tilde Y$, we have that

$$\dim_p\tilde Z=\dim\mathcal{O}^{\rm an}_{\tilde Z,p}<\dim\mathcal{O}^{\rm an}_{\tilde Y,p}=\dim_p\tilde Y.$$

Since $Z=\tilde Z\cap Y$, we have that T-$\dim_p Z\le\dim_p\tilde Z$ for all $p\in Z$. Since Z is closed in Y, we have that Z is nowhere dense in Y. ☐

Lemma 9.3. *Suppose that $\varphi:Y\to X$ is a morphism of connected smooth real analytic varieties and $\tilde\varphi:\tilde Y\to\tilde X$ is a complexification of φ (with $\tilde Y$ and $\tilde X$ smooth). Then φ is regular if and only if $\tilde\varphi$ is regular.*

Proof. Suppose that $\tilde\varphi$ is regular. Let $n=\dim\tilde X$. Then the closed analytic subspace

$$\tilde Z=\{\tilde q\in\tilde Y\mid \mathrm{rank}(d\tilde\varphi_{\tilde q})<n\}$$

is a proper subset of $\tilde Y$. Suppose that $\varphi:Y\to X$ is not regular. Then

$$Y=\{q\in Y\mid\mathrm{rank}(d\varphi_q)<n\}=\tilde Z\cap Y,$$

a contradiction to Lemma 9.2.

A simpler argument shows that if φ is regular then $\tilde\varphi$ is regular. ☐

Proposition 9.4. *Suppose that $\varphi:Y\to X$ is a morphism of reduced real analytic spaces with complexification $\tilde\varphi:\tilde Y\to\tilde X$, such that there are auto conjugations $\sigma:\tilde X\to\tilde X$ and $\tau:\tilde Y\to\tilde Y$ such that $\tilde\varphi\tau=\sigma\tilde\varphi$. Let $e\in\mathcal{E}_{\tilde Y}$ be an étoile over $\tilde Y$. Then there exists a*

commutative diagram of morphisms

$$\begin{array}{ccc} \tilde Y_e & \overset{\tilde\varphi_e}{\to} & \tilde X_e\\ \tilde\delta\downarrow & & \downarrow\tilde\gamma\\ \tilde Y & \overset{\tilde\varphi}{\to} & \tilde X\end{array}$$

such that $\tilde\delta\in e$, $\tilde Y_e$ and $\tilde X_e$ are smooth analytic spaces, there exists a closed analytic sub manifold $\tilde Z_e$ of $\tilde X_e$ such that $\tilde\varphi_e(\tilde Y_e)\subset\tilde Z_e$ and the induced analytic map $\tilde\varphi_e:\tilde Y_e\to\tilde Z_e$ is regular. Further, there exists a nowhere dense closed analytic subspace $\tilde F_e$ of $\tilde X_e$ such that $\tilde X_e\setminus\tilde F_e\to\tilde X$ is an open embedding and $\tilde\varphi_e^{-1}(\tilde F_e)$ is nowhere dense in $\tilde Y_e$.

There exist auto conjugations $\sigma_e:\tilde X_e\to\tilde X_e$ and $\tau_e:\tilde Y_e\to\tilde Y_e$ which are compatible with the diagram. We have that $\sigma_e(\tilde Z_e)=\tilde Z_e$ and $\sigma_e(\tilde F_e)=\tilde F_e$.

Further, we have a factorization of $\tilde\delta$ as

$$\tilde Y_e=W_s\overset{\beta_{s-1}}{\to}W_{s-1}\to\cdots\to W_1\overset{\beta_0}{\to}W_0=\tilde Y$$

where each β_i is a local blow up (U_i,E_i,β_i) where E_i is a smooth sub variety of U_i and there are auto conjugations $\tau_i:W_i\to W_i$ such that $\beta_i\tau_{i+1}=\tau_i\beta_i$ for all i. We have that $\tau_0=\tau$ and $\tau_s=\tau_e$ and $\tau_i(U_i)=U_i$ and $\tau_i(E_i)=E_i$ for all i. Further, either the center e_{W_i} of e on W_i is a real point ($\tau_i(e_{W_i})=e_{W_i}$) or $\tau_i(e_{W_i})\ne e_{W_i}$ and U_i is the disjoint union of two open subsets S_i and $\tau_i(S_i)$ which are respective open neighborhoods of e_{W_i} and $\tau_i(e_{W_i})$.

We also have a factorization of $\tilde\gamma$ by

$$\tilde X_e=Z_r\overset{\alpha_{r-1}}{\to}Z_{r-1}\to\cdots\to Z_1\overset{\alpha_0}{\to}Z_0=\tilde X$$

where each α_i is a local blow up (V_i,H_i,α_i) where H_i is a smooth sub variety of V_i, and there are auto conjugations $\sigma_i:Z_i\to Z_i$ such that $\alpha_i\sigma_{i+1}=\sigma_i\alpha_i$, $\sigma_i(V_i)=V_i$ and $\sigma_i(H_i)=H_i$. Further either $q_i:=\alpha_i\cdots\alpha_{r-1}\tilde\varphi_e(e_{\tilde Y_e})$ is a real point ($\sigma_i(q_i)=q_i$) or $\sigma_i(q_i)\ne q_i$ and V_i is the disjoint union of two open subsets T_i and $\sigma_i(T_i)$ which are respective open neighborhoods of q_i and $\sigma_i(q_i)$. We have that $\sigma_0=\sigma$ and $\sigma_r=\sigma_e$.

We obtain the conclusions of Proposition 9.4, by modifying the proof of Proposition 3.5, using Corollary 8.11 and Remark 9.1.

Proposition 9.5. *Suppose that $\varphi:Y\to X$ is a regular morphism of real analytic manifolds, E_Y is a SNC divisor on Y with complexification $\tilde\varphi:\tilde Y\to\tilde X$ where $\tilde Y$ and $\tilde X$ are complex analytic manifolds and complexification $E_{\tilde Y}$ of E_Y which is a SNC divisor on $\tilde Y$. Let e be an étoile over $\tilde Y$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \tilde Y_e & \overset{\tilde\varphi_e}{\to} & \tilde X_e\\ \tilde\pi_e\downarrow & & \downarrow\tilde\lambda_e\\ \tilde Y & \overset{\tilde\varphi}{\to} & \tilde X\end{array}$$

of regular complex analytic morphisms such that the vertical arrows are products of local blow ups of nonsingular analytic subvarieties such that $\tilde Y_e\to\tilde Y\in e$.

The vertical arrows have factorizations by sequences of local blow ups

$$\begin{array}{ccccccc} \tilde Y_e=W_t & \overset{\beta_t}{\to} & \cdots & \to & W_1 & \overset{\beta_1}{\to} & W_0=\tilde Y\\ \downarrow & & & & \downarrow & & \downarrow\\ \tilde X_e=V_t & \overset{\alpha_t}{\to} & \cdots & \to & V_1 & \overset{\alpha_1}{\to} & V_0=\tilde X\end{array}\tag{60}$$

with complex auto conjugations of the W_i and V_i which are compatible with the above diagram, so that taking the invariants of these auto conjugations, we have an induced diagram of regular real analytic morphisms

$$\begin{array}{ccc} Y_e & \overset{\varphi_e}{\rightarrow} & X_e \\ \pi_e \downarrow & & \downarrow \lambda_e \\ Y & \overset{\varphi}{\rightarrow} & X \end{array}$$

such that the vertical arrows are products of local blow ups of nonsingular real analytic subvarieties. The auto conjugations of W_i induce auto conjugations of the preimage of $\tilde{E}_Y$ on W_i.

Either all e_{W_i} (and e_{V_i}) are real points in the diagram (60), and φ_e and $\tilde{\varphi}_e$ are monomial morphisms for toroidal structures O_e on Y_e with complexification $\tilde{O}_e$ on $\tilde{Y}_e$ or $e_{\tilde{Y}_e}$ is not a real point, and Y_e is the empty set.

Further, we have that $\tilde{\pi}_e^*(E_{\tilde{Y}})$ is an effective divisor supported on $\tilde{O}_e$ and the restriction of $\tilde{\pi}_e$ to $\tilde{Y}_e \setminus \tilde{O}_e$ is an open embedding. Also, $\pi^*(E_Y)$ is an effective divisor supported on O_e, and the restriction of π_e to $Y_e \setminus O_e$ is an open embedding.

Proof. We inductively construct the diagram (60) of local blow ups as in the proof of Proposition 8.10, with the following differences. If after construction of the local blow up $W_i \rightarrow W_{i-1}$ we find that e_{W_i} is not a real point then we take a neighborhood U of e_{W_i} which contains no real points and set W_{i+1} to be the (disjoint) union of U and $\sigma(U)$ where σ is the auto conjugation of W_i. We then terminate the algorithm, setting $\tilde{Y}_e = W_{i+1}$ and $\tilde{X}_e = V_i$.

In our inductive construction of (60), as long as e_{W_j} are real points for $j \leq i$, the sequences of local blow ups in (60) are complexifications of sequences of real local blow ups of nonsingular real analytic subvarieties. This follows from the algorithms of Proposition 8.10, as we then work within the rings

$$\begin{array}{ccc} \mathbb{R}\{\{x_1,\ldots,x_m\}\} & \rightarrow & \mathbb{R}\{\{y_1,\ldots,y_n\}\} \\ \downarrow & & \downarrow \\ \mathbb{R}[[x_1,\ldots,x_m]] & \rightarrow & \mathbb{R}[[y_1,\ldots,y_n]] \end{array}$$

instead of in the corresponding complexifications of these rings.

The only modification which needs to be made in the algorithm (since we assume all centers of e are real) is that a little more care is needed when taking roots of unit series. For instance, in Lemma 5.6, we must insist that the constant term of the unit γ is positive. This leads to the introduction of factors of ± 1 in the equations of Lemmas 5.7, 6.3 and 6.4. To preserve the monomial form (15), we may have to replace some of the $y_j(1)$ with their negatives $-y_j(1)$ and some of the $x_i(1)$ with their negatives $-x_i(1)$. We also need the conclusions of Lemma 9.3. ☐

Proposition 9.6. *Suppose that $\varphi : Y \rightarrow X$ is a morphism of reduced real analytic spaces and $A \subset Y$ is a closed analytic subspace of Y, with complexification $\tilde{\varphi} : \tilde{Y} \rightarrow \tilde{X}$ of φ and complexification $\tilde{A} \subset \tilde{Y}$ of A. Let e be an étoile over $\tilde{Y}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \tilde{Y}_e & \overset{\tilde{\varphi}_e}{\rightarrow} & \tilde{X}_e \\ \tilde{\beta} \downarrow & & \downarrow \tilde{\alpha} \\ \tilde{Y} & \overset{\tilde{\varphi}}{\rightarrow} & \tilde{X} \end{array}$$

of complex analytic morphisms such that $\tilde{Y}_e$ and $\tilde{X}_e$ are smooth analytic spaces, $\tilde{\beta}\in e$ and we have a factorization of $\tilde{\beta}$ as

$$\tilde{Y}_e = W_s \overset{\beta_{s-1}}{\to} W_{s-1} \to \cdots \to W_1 \overset{\beta_0}{\to} W_0 = \tilde{Y}$$

where each β_i is a local blow up (U_i, E_i, β_i) where E_i is a smooth sub variety of U_i and there are auto conjugations $\tau_i : W_i \to W_i$ such that $\beta_i\tau_{i+1} = \tau_i\beta_i$ for all i. We have that $\tau_0 = \tau$ and $\tau_s = \tau_e$ and $\tau_i(U_i) = U_i$ and $\tau_i(E_i) = E_i$ for all i. Further, either the center e_{W_i} of e on W_i is a real point ($\tau_i(e_{W_i}) = e_{W_i}$) or $\tau_i(e_{W_i}) \neq e_{W_i}$ and U_i is the disjoint union of two open subsets S_i and $\tau_i(S_i)$ which are respective open neighborhoods of e_{W_i} and $\tau_i(e_{W_i})$.

We also have a factorization of $\tilde{\alpha}$ by

$$\tilde{X}_e = Z_r \overset{\alpha_{r-1}}{\to} Z_{r-1} \to \cdots \to Z_1 \overset{\alpha_0}{\to} Z_0 = \tilde{X}$$

where each α_i is a local blow up (V_i, H_i, α_i) where H_i is a smooth sub variety of V_i, and there are auto conjugations $\sigma_i : Z_i \to Z_i$ such that $\alpha_i\sigma_{i+1} = \sigma_i\alpha_i$, $\sigma_i(V_i) = V_i$ and $\sigma_i(H_i) = H_i$. Further either $q_i := \alpha_i \cdots \alpha_{r-1}\tilde{\varphi}_e(e_{\tilde{Y}_e})$ is a real point ($\sigma_i(q_i) = q_i$) or $\sigma_i(q_i) \neq q_i$ and V_i is the disjoint union of two open subsets T_i and $\sigma_i(T_i)$ which are respective open neighborhoods of q_i and $\sigma_i(q_i)$. We have that $\sigma_0 = \sigma$ and $\sigma_r = \sigma_e$. Further, there exists a nowhere dense closed analytic subspace $\tilde{F}_e$ of $\tilde{X}_e$ such that $\sigma_e(\tilde{F}_e) = \tilde{F}_e$, $\tilde{X}_e\backslash\tilde{F}_e \to \tilde{X}$ is an open embedding and $\tilde{\varphi}_e^{-1}(\tilde{F}_e)$ is nowhere dense in $\tilde{Y}_e$.

Taking the invariants of these auto conjugations, we have an induced diagram of real analytic morphisms

$$\begin{array}{ccc} Y_e & \overset{\varphi_e}{\to} & X_e \\ \beta\downarrow & & \downarrow\alpha \\ Y & \overset{\varphi}{\to} & X \end{array}$$

such that the vertical arrows are products of local blow ups of nonsingular real analytic subvarieties. Either all e_{W_i} (and e_{V_i}) are real points and φ_e and $\tilde{\varphi}_e$ are monomial morphisms for toroidal structures O_e on Y_e at $e_{\tilde{Y}_e}$ with complexification $\tilde{O}_e$ on $\tilde{Y}_e$ or $e_{\tilde{Y}_e}$ is not a real point, and Y_e is the empty set.

Further, either the preimage of $\tilde{A}$ in $\tilde{Y}_e$ is equal to $\tilde{Y}_e$ or $\mathcal{I}_{\tilde{A}}\mathcal{O}^{\rm an}_{\tilde{Y}_e} = \mathcal{O}^{\rm an}_{\tilde{Y}_e}(-G)$ where $\mathcal{I}_{\tilde{A}}$ is the ideal sheaf in $\mathcal{O}^{\rm an}_{\tilde{Y}}$ of the analytic subspace $\tilde{A}$ of $\tilde{Y}$, $\tilde{G}$ is an effective divisor which is supported on $\tilde{O}_e$ and $\tilde{Y}_e \setminus \tilde{O}_e \to \tilde{Y}$ is an open embedding. We have that $\tau_e(G) = G$.

Suppose that $e_{\tilde{Y}_e}$ is real. Then $F_e = \tilde{F}_e \cap X_e$ is nowhere dense in X_e, $X_e \setminus F_e \to X$ is an open embedding and $\varphi_e^{-1}(F_e)$ is nowhere dense in Y_e.

We obtain Proposition 9.6 by arguing as in the proof of Theorem 8.12, using Proposition 9.4, Remark 9.1, Proposition 9.5, Lemma 9.3 and Lemma 9.2.

We have the following theorem, which generalizes Theorem 8.13 to a real analytic morphism from a real analytic manifold.

Theorem 9.7. *Suppose that Y is a real analytic manifold, X is a reduced real analytic space, $\varphi : Y \to X$ is a real analytic morphism, A is a closed analytic subspace of Y and $p \in Y$. Then there exists a finite number t of commutative diagrams of real analytic morphisms*

$$\begin{array}{ccc} Y_i & \overset{\varphi_i}{\to} & X_i \\ \beta_i\downarrow & & \downarrow\alpha_i \\ Y & \overset{\varphi}{\to} & X \end{array}$$

for $1 \leq i \leq t$ such that each β_i and α_i is a finite product of local blow ups of nonsingular analytic sub varieties, Y_i and X_i are smooth analytic spaces and φ_i is a monomial analytic morphism for a toroidal structure O_i on Y_i. Either the preimage of A in Y_i is Y_i, or $\mathcal{I}_A\mathcal{O}_{Y_i} = \mathcal{O}_{Y_i}(-G_i)$ where $\mathcal{I}_A$ is the ideal sheaf in $\mathcal{O}_Y^{\mathrm{an}}$ of the analytic subspace A of Y, G_i is an effective divisor which is supported on O_i, and has the further property that the restriction $(Y_i \setminus O_i) \to Y$ is an open embedding. Further, there exist compact subsets K_i of Y_i such that $\cup_{i=1}^t \beta_i(K_i)$ is a compact neighborhood of p in Y. There exist nowhere dense closed analytic subspaces F_i of X_i such that $X_i \setminus F_i \to X$ are open embeddings and $\varphi_i^{-1}(F_i)$ is nowhere dense in Y_i.

Proof. Let $\tilde{\varphi} : \tilde{Y} \to \tilde{X}$ be a complexification of φ such that $\tilde{Y}$ is nonsingular.

Suppose that $e \in \mathcal{E}_{\tilde{Y}}$ (the voûte étoilée and the notation used in this proof are reviewed in Section 3). Then we may construct a diagram satisfying the conclusions of Proposition 9.6

$$\begin{array}{ccc} \tilde{Y}_e & \overset{\tilde{\varphi}_e}{\to} & \tilde{X}_e \\ \tilde{\beta}_e \downarrow & & \downarrow \tilde{\alpha}_e \\ \tilde{Y} & \overset{\tilde{\varphi}}{\to} & \tilde{X} \end{array}$$

with real part

$$\begin{array}{ccc} Y_e & \overset{\varphi_e}{\to} & X_e \\ \beta_e \downarrow & & \downarrow \alpha_e \\ Y & \overset{\varphi}{\to} & X. \end{array}$$

(We can have $Y_e = \emptyset$).

Let $\tilde{C}_e$ be an open relatively compact neighborhood of $e_{\tilde{Y}_e}$ in $\tilde{Y}_e$ on which the auto conjugation acts. Let $\overline{\beta}_e : \tilde{C}_e \to \tilde{Y}$ be the induced map.

Let K be a compact neighborhood of p in $\tilde{Y}$ and $K' = P_{\tilde{Y}}^{-1}(K)$. The set K' is compact since $P_{\tilde{Y}}$ is proper (Theorem 3.4 [43]). The open sets $\mathcal{E}_{\overline{\beta}_e}$ for $e \in K'$ (defined in equation (6)) give an open cover of K', so there is a finite subcover, which we index as $\mathcal{E}_{\overline{\beta}_{e_1}}, \ldots, \mathcal{E}_{\overline{\beta}_{e_t}}$. Let K_i be the closure of $\tilde{C}_{e_i}$ in $\tilde{Y}_{e_i}$ which is compact. Since $P_{\tilde{Y}}$ is surjective and continuous, we have inclusions of compact sets $p \in K \subset \cup_{i=1}^t \tilde{\beta}_{e_i}(K_i)$. Since $\tilde{Y}$ is nonsingular and each $\tilde{\beta}_{e_i}$ is a (finite) product of local blow ups of proper sub varieties, if $\tilde{H}_{e_i}$ is the union of the preimages on $\tilde{Y}_{e_i}$ of these centers, then $\tilde{H}_{e_i}$ is a nowhere dense closed analytic subspace of $\tilde{Y}_{e_i}$ and $\tilde{\beta}_{e_i}$ is an open embedding of $\tilde{Y}_{e_i} \setminus \tilde{H}_{e_i}$ into $\tilde{Y}$.

Suppose that $q \in Y$. Then $\text{T-}\dim_q Y = \dim_q \tilde{Y}$ since Y is a manifold (Section 3 and Section 5 of [42]). Suppose i satisfies $1 \leq i \leq t$. The set $\tilde{H}_{e_i} \cap K_i$ is compact and $\tilde{\beta}_{e_i}(\tilde{H}_{e_i} \cap K_i)$ is compact. Let $M_i = \tilde{\beta}_{e_i}(\tilde{H}_{e_i} \cap K_i) \cap Y$. Suppose $q \in M_i$. Then

$$\dim_q \tilde{\beta}_{e_i}(\tilde{H}_{e_i} \cap K_i) < \dim \tilde{Y}$$

by Theorem 1, page 254 [47] or Corollary 1, page 255 [47]. Thus

$$\text{T-}\dim_q M_i \leq \dim_q \tilde{\beta}_{e_i}(\tilde{H}_{e_i} \cap K_i) < \dim_q \tilde{Y} = \text{T-}\dim_q Y.$$

Since M_i is compact, we have that M_i is nowhere dense in Y.

Let $K^* = K \cap Y$ which is a compact neighborhood of p in Y. Let $p' \in K^* \setminus \cup_{i=1}^t \tilde{\beta}_{e_i}(\tilde{H}_{e_i} \cap K_i)$. Then there exist i and $e \in \mathcal{E}_{\overline{\beta}_{e_i}}$ such that $e_{\tilde{Y}} = p'$ and $p_i = e_{\tilde{Y}_{e_i}} \in K_i \setminus \tilde{H}_{e_i} \subset \tilde{Y}_{e_i}$. Since $p_i \notin \tilde{H}_{e_i}$, $\tilde{\beta}_{e_i}$ is an open embedding near p_i, and since p' is real, $p_i \in Y_{e_i}$ is real. Thus $p' \in \beta_{e_i}(K_i \cap Y_{e_i})$. We thus have that the set $K^* \setminus \cup_{i=1}^t \tilde{\beta}_{e_i}(\tilde{H}_{e_i} \cap K_i)$, which we have

shown is dense in K^*, is contained in the compact set $\cup_{i=1}^t \beta_{e_i}(K_i \cap Y_{e_i})$. Thus its closure K^* is contained in $\cup_{i=1}^t \beta_{e_i}(K_i \cap Y_{e_i})$, giving the conclusions of Theorem 9.7. □

Theorems 1.4 and 1.7 of the introduction follow from Theorem 9.7.

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